A REMARK ON KRONECKER’S THEOREM ON FORMS

HARLEY FLANDERS

The Kronecker theorem on forms over an integral domain is a consequence of integral closure. We refer the reader to [1]1 for a proof and other references. We have failed to find in the literature a statement of the converse of this result and consequently shall here prove that integral closure is a consequence of a relatively weak form [2] of Kronecker’s theorem.

Let \( \mathcal{O} \) be an integral domain which has the following property: Whenever each coefficient of the product of a linear polynomial \( f(X) = a_0X + a_1 \) of \( \mathcal{O}[X] \) by an arbitrary polynomial \( g(X) = b_0X^n + \cdots + b_n \) of \( \mathcal{O}[X] \) is divisible by an element \( c \) of \( \mathcal{O} \), then \( a_0b_0 \) is divisible by \( c \). Then \( \mathcal{O} \) is integrally closed in its quotient field \( k \).

To prove this, we let \( \alpha = u/v \in k \), \( u, v \in \mathcal{O} \), and assume \( \alpha \) is integral over \( \mathcal{O} \). Thus \( h(\alpha) = 0 \) where

\[
h(X) = X^m + c_1X^{m-1} + \cdots + c_m
\]

with all \( c_j \in \mathcal{O} \). Clearly

\[
h(X) = (X - \alpha)(X^{m-1} + \beta_1X^{m-2} + \cdots + \beta_{m-1})
\]

with \( \beta_j \in k \). We select \( w \in \mathcal{O} \) such that \( w\beta_j \in \mathcal{O} \) for all \( j \) and have

\[
vwh(X) = (vX - u)(wX^{m-1} + w\beta_1X^{m-2} + \cdots + w\beta_{m-1}).
\]

Since \( vw \) divides each coefficient of the left-hand side of this equation, it follows from the hypothesis that \( vw \) divides \( uw \), and hence \( v \) divides \( u \). This implies that \( \alpha \) is in \( \mathcal{O} \), which completes the proof.

REFERENCES


California Institute of Technology

Received by the editors August 6, 1951.

1 Numbers in brackets refer to the references at the end of the paper.

197

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use