of our map is a maximal ideal and we see that Statements S2 and S3 are violated. This concludes the proof of our theorem.

Reference


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ON ORDERED SKEW FIELDS

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In this paper we shall give a necessary and sufficient condition that a skew field can be ordered; moreover, that the ordering of an ordered skew field \( K \) can be extended to an ordering of \( L \), \( L \) being a given extension of \( K \). The first of these two results generalizes to skew fields a theorem of E. Artin and O. Schreier \[1\], according to which a commutative field can be ordered if and only if it is formally real. The second result generalizes in the same sense a recent theorem of J. P. Serre \[2\].

Our considerations are based on the following definition.

**Definition.** A skew field is said to be ordered if in its multiplicative group a subgroup of index 2 is marked out which is also closed under addition.

Hence a skew field can be ordered if and only if its multiplicative group has a subgroup of index 2 which is also closed under addition.

We shall now prove the following theorem.

**Theorem 1.** A skew field \( K \) can be ordered if and only if \(-1\) cannot be represented as a sum of elements of the form

\[
2 \quad 2 \\
\quad a_1 a_2 \cdots a_k \quad (a_i \in K, i = 1, 2, \ldots, k).
\]

**Remark.** This property can be considered as a generalization of the notion "formally real" to the case of skew fields.

The necessity of the condition in Theorem 1 is obvious. In order to prove its sufficiency we consider a skew field \( K \) in which \(-1\) cannot be represented as a sum of elements (1). We shall show that the

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1 Numbers in brackets refer to the bibliography at the end of the paper.
On Ordered Skew Fields

Multiplicative group $K^*$ of $K$ has a subgroup of index 2 which is also closed under addition.

Let $S$ be the subset of all (finite) sums of elements (1) in $K$ with every $a_i \neq 0$. Clearly $0 \in S$, for in the contrary case we should have a relation

$$-a_1^2a_2^2 \cdots a_k^2 = b_1b_2 \cdots b_1^2 + \cdots$$

from which would follow, by multiplication on the right by $a_1^{-2}$, $\cdots$, $a_k^{-2}$, that $-1 \in S$ in contradiction to our hypothesis. On the other hand, one can see immediately that

- $s \in S$, $s' \in S$ imply $ss' \in S$ and $s + s' \in S$,
- $s \in S$ implies $s^{-1} = s \cdot s^{-2} \in S$,
- $s \in S$, $z \in K^*$ imply $z^{-1}s \in S$.

Hence $S$ is a proper invariant subgroup of $K^*$ which is closed under addition. The order of each element ($\neq 1$) in $K^*/S$ being 2, $K^*/S$ is abelian. Consequently any subgroup $P$ of $K^*$ which contains $S$ is invariant in $K^*$.

Now we define $P$ as a maximal subgroup of $K^*$ for which

(2) \quad $S \subseteq P$, \quad $-1 \notin P$, and $P$ is closed under addition.

The existence of such a group $P$ follows immediately from Zorn's lemma. We have only to show that the decomposition

(3) \quad $K^* = P \cup \{-1\}P$

holds. Suppose (3) is not true. Then there exists an element $d$ such that

(4) \quad $d \in K^*$, \quad $d \in P$, \quad $-d \notin P$.

Consider the set $P'$ of all elements

$$u + vd \quad (u, v \in \{P, 0\} \text{ but not } u = v = 0).$$

Then, by (4), $P'$ contains $P$ as a proper subset. On the other hand we shall show that $P'$ is a subgroup of $K^*$ having the properties (2) (with $P'$ instead of $P$). This is a contradiction to the maximal property of $P$, which will complete the proof.

First we show that $0 \in P'$. Indeed, by the exclusion of $u = v = 0$, $u + vd = 0$ would imply that $v \neq 0$ and hence that $-d = v^{-1}u \in P$, in contradiction to (4). Moreover, if $u_1 + v_1d$ and $u_2 + v_2d$ are arbitrary elements of $P'$, we have

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(5) \((u_1 + v_1 d)(u_2 + v_2 d) = (u_1 u_2 + v_1 v_2 d) + (u_1 v_2 d + v_1 d u_2)\).

But since \(P\) is an invariant subgroup of \(K^*\), \(d v_1 = v_1 d, d u_2 = u_2 d\) hold with suitable elements \(u_1', v_1 \in P\), so that (5) is an element of \(P'\). If \(u + v d \in P'\), we obtain
\[
(u + v d)^{-1} = (u + v d)(u + v d)^{-2} \in P'.
\]

Hence \(P'\) is a group which is obviously closed under addition. Finally, \(-1 \in P'\) for \(u + v d = -1\) would imply (on account of \(v \neq 0\)) that \(-d = v^{-1}(u + 1) \in P\). This completes the proof.

In an analogous manner we prove the following theorem.

**Theorem 2.** Let \(L\) be an extension of the ordered skew field \(K\). The ordering of \(K\) can be extended to an ordering of \(L\) if and only if \(-1\) cannot be represented as a sum of elements

\[
(p_1 u_1^2 \cdots p_k u_k^2) \quad (p_i \in K, p_i > 0, u_i \in L, i = 1, 2, \ldots, k).
\]

**Remark.** Theorem 1 is the special case of Theorem 2 in which \(K\) is the prime field of characteristic zero. However, this special case seemed of sufficient interest to warrant an independent proof. Only a few remarks are now necessary to prove Theorem 2 since the proof follows the same general pattern as that of Theorem 1.

The necessity of the condition in Theorem 2 is obvious. In order to prove its sufficiency we define the subset \(U\) of \(L\) as the set of all \((\text{finite})\) sums of elements (6) with every \(m \neq 0\). One can show as above that \(U\) is a subgroup of the multiplicative group \(L^*\) of \(L\). That, e.g.,
\[
0 \in U
\]
follows from the fact that a relation
\[
-1 = p_1 v_1^2 \cdots p_k v_k^2 = p_1' v_1^2 \cdots p_k' v_k^2 + \cdots
\]
would imply that
\[
1 = p_1' v_1^2 \cdots p_k' v_k^2 (1 - u_k^{-2}) (p_k' u_{k-1}^{-2}) \cdots (p_1' 1^2) + \cdots,
\]
which is impossible.

Moreover \(U\) is an invariant subgroup of \(L^*\). This follows from the fact that
\[
p \in K, \quad p > 0, \quad z \in L^*
\]
imply that
\[
z^{-1} p z = z^{-2} z p z p^{-1} = p_1' v_1^2 p_2' v_2^2 p_3' + \cdots
\]
with \(p_1' = 1, v_1 = z^{-1}, p_2' = 1, v_2 = z p, p_3' = p^{-1}\).
From the fact that each element \((\neq 1)\) of \(L^*/U\) is of order 2, we infer as above that any subgroup \(Q\) of \(L^*\) containing \(U\) is invariant in \(L^*\).

Now we define \(Q\) as a maximal subgroup of \(L^*\) for which \(U \subseteq Q\), \(-1 \in Q\), and \(Q\) is closed under addition. Then one can show as above that \(Q\) is a subgroup of index 2 of \(L^*\). Since all positive elements of \(K\) are contained in \(U\) and consequently in \(Q\), the theorem is proved.


Bibliography
