of our map is a maximal ideal and we see that Statements S2 and S3 are violated. This concludes the proof of our theorem.

Reference


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ON ORDERED SKEW FIELDS

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In this paper we shall give a necessary and sufficient condition that a skew field can be ordered; moreover, that the ordering of an ordered skew field \( K \) can be extended to an ordering of \( L \), \( L \) being a given extension of \( K \). The first of these two results generalizes to skew fields a theorem of E. Artin and O. Schreier [1], according to which a commutative field can be ordered if and only if it is formally real. The second result generalizes in the same sense a recent theorem of J. P. Serre [2].

Our considerations are based on the following definition.

Definition. A skew field is said to be ordered if in its multiplicative group a subgroup of index 2 is marked out which is also closed under addition.

Hence a skew field can be ordered if and only if its multiplicative group has a subgroup of index 2 which is also closed under addition.

We shall now prove the following theorem.

Theorem 1. A skew field \( K \) can be ordered if and only if \(-1\) cannot be represented as a sum of elements of the form

\[
(1) \quad a_1^2 a_2^2 \cdots a_k^2 \quad (a_i \in K, i = 1, 2, \ldots, k).
\]

Remark. This property can be considered as a generalization of the notion "formally real" to the case of skew fields.

The necessity of the condition in Theorem 1 is obvious. In order to prove its sufficiency we consider a skew field \( K \) in which \(-1\) cannot be represented as a sum of elements (1). We shall show that the
multiplicative group $K^*$ of $K$ has a subgroup of index 2 which is also closed under addition.

Let $S$ be the subset of all (finite) sums of elements (1) in $K$ with every $a_i \neq 0$. Clearly $0 \in S$, for in the contrary case we should have a relation

$$-a_1^2 \cdots -a_k^2 = b_1^2 \cdots b_l^2 + \cdots$$

from which would follow, by multiplication on the right by $a_1^{-2}, \ldots, a_k^{-2}$, that $-1 \in S$ in contradiction to our hypothesis. On the other hand, one can see immediately that

- $s \in S$, $s' \in S$ imply $ss' \in S$ and $s + s' \in S$,
- $s \in S$ implies $s^{-1} = s \cdot s^{-2} \in S$,
- $s \in S$, $z \in K^*$ imply $z^{-1}sz \in S$.

Hence $S$ is a proper invariant subgroup of $K^*$ which is closed under addition. The order of each element ($\neq 1$) in $K^*/S$ being 2, $K^*/S$ is abelian. Consequently any subgroup $P$ of $K^*$ which contains $S$ is invariant in $K^*$.

Now we define $P$ as a maximal subgroup of $K^*$ for which

$$S \subseteq P, \quad -1 \notin P, \text{ and } P \text{ is closed under addition.} \tag{2}$$

The existence of such a group $P$ follows immediately from Zorn’s lemma. We have only to show that the decomposition

$$K^* = P \cup (-1)P \tag{3}$$

holds. Suppose (3) is not true. Then there exists an element $d$ such that

$$d \in K^*, \quad d \in P, \quad -d \notin P. \tag{4}$$

Consider the set $P'$ of all elements

$$u + vd \quad (u, v \in \{P, 0\} \text{ but not } u = v = 0).$$

Then, by (4), $P'$ contains $P$ as a proper subset. On the other hand we shall show that $P'$ is a subgroup of $K^*$ having the properties (2) (with $P'$ instead of $P$). This is a contradiction to the maximal property of $P$, which will complete the proof.

First we show that $0 \notin P'$. Indeed, by the exclusion of $u = v = 0$, $u + vd = 0$ would imply that $v \neq 0$ and hence that $-d = v^{-1}u \in P$, in contradiction to (4). Moreover, if $u_1 + v_1d$ and $u_2 + v_2d$ are arbitrary elements of $P'$, we have
(5) \((u_1 + v_1d)(u_2 + v_2d) = (u_1u_2 + v_1d\ u_2d) + (u_1v_2d + v_1du_2).\)

But since \(P\) is an invariant subgroup of \(K^*,\) \(du_2 = v_2d, du_1 = u_1d\) hold with suitable elements \(u_1', v_1' \in P,\) so that (5) is an element of \(P'.\) If \(u + vd \in P',\) we obtain

\[(u + vd)^{-1} = (u + vd)(u + vd)^{-2} \in P'.\]

Hence \(P'\) is a group which is obviously closed under addition. Finally, \(-1 \in P'\) for \(u + vd = -1\) would imply (on account of \(v \neq 0\)) that \(-d = v^{-1}(u + 1) \in P.\) This completes the proof.

In an analogous manner we prove the following theorem.

**Theorem 2.** Let \(L\) be an extension of the ordered skew field \(K.\) The ordering of \(K\) can be extended to an ordering of \(L\) if and only if \(-1\) cannot be represented as a sum of elements

\[(6) \ p_1u_1^2 \cdots p_ku_k^2 \quad (p_i \in K, p_i > 0, u_i \in L, i = 1, 2, \cdots, k).\]

**Remark.** Theorem 1 is the special case of Theorem 2 in which \(K\) is the prime field of characteristic zero. However, this special case seemed of sufficient interest to warrant an independent proof. Only a few remarks are now necessary to prove Theorem 2 since the proof follows the same general pattern as that of Theorem 1.

The necessity of the condition in Theorem 2 is obvious. In order to prove its sufficiency we define the subset \(U\) of \(L\) as the set of all (finite) sums of elements (6) with every \(u_i \neq 0.\) One can show as above that \(U\) is a subgroup of the multiplicative group \(L^*\) of \(L.\) That, e.g., \(0 \in U\) follows from the fact that a relation

\[-p_1u_1^2 \cdots p_ku_k^2 = p'_1v_1^2 \cdots p'_k v_k^2 + \cdots\]

would imply that

\[-1 = p'_1v_1^2 \cdots p'_k v_k^2 (1 \cdot u_k^{-2})(p_k^{-1} u_k^{-2}) \cdots (p_1^{-1} u_1^{-2}) + \cdots,\]

which is impossible.

Moreover \(U\) is an invariant subgroup of \(L^*.\) This follows from the fact that

\[p \in K, \quad p > 0, \quad z \in L^*\]

imply that

\[z^{-1} p z = z^{-2} z p z p^{-1} = p'_1v_1p_1v_2p_2^2 \cdots \]

with \(p'_1 = 1, v_1 = z^{-1}, p'_2 = 1, v_2 = z p, p'_3 = p^{-1}.\)
From the fact that each element \((\neq 1)\) of \(L^*/U\) is of order 2, we infer as above that any subgroup \(Q\) of \(L^*\) containing \(U\) is invariant in \(L^*\).

Now we define \(Q\) as a maximal subgroup of \(L^*\) for which \(U \subseteq Q\), \(-1 \in Q\), and \(Q\) is closed under addition. Then one can show as above that \(Q\) is a subgroup of index 2 of \(L^*\). Since all positive elements of \(K\) are contained in \(U\) and consequently in \(Q\), the theorem is proved.


**Bibliography**


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