ON ORDERED DOMAINS OF INTEGRITY

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In a recent paper, T. Szele proved that a division ring D is orderable if and only if the additive and multiplicative semigroup S generated by the nonzero squares of elements of D does not contain the zero element of D. The present paper extends this result to a domain of integrity K.

Let us denote by $K^*$ the set of nonzero elements of K. The domain of integrity K is said to be orderable if and only if there exists an additive and multiplicative semigroup $P$ (the positive elements) contained in $K^*$ such that $K^* = P \cup (-P)$.

If K does not have a unit element, then there exists a unique minimal domain of integrity $\overline{K}$ having a unit element and containing K. It is not too difficult to show that K is orderable if and only if $\overline{K}$ is orderable. For this reason, we assume henceforth that K has a unit element.

An element $a$ of $K^*$ is called even if there exist elements $a_1, \ldots, a_n$ in $K^*$ such that $a$ is a product of the $2n$ elements $a_1, \ldots, a_n, a_1, \ldots, a_n$ in some order. We denote by $\mathcal{S}$ the additive semigroup generated by the even elements of $K^*$. The theorem we wish to prove is as follows.

**Theorem.** The domain of integrity K is orderable if and only if $\mathcal{S} \subseteq K^*$.

The proof of this theorem will come after a few preliminary remarks. For any subset $A$ of K, an element $b$ of K is called even over $A$ if there exist elements $a_1, \ldots, a_n$ of A and $k_1, \ldots, k_m$ of $K^*$ such that $b$ is a product of the $2m+n$ elements $a_1, \ldots, a_n, k_1, \ldots, k_m, k_1, \ldots, k_m$ in some order. We denote by $\mathcal{E}$ the set of all elements of K even over $A$. Evidently $\mathcal{E}$ is a multiplicative semigroup containing $A$, and $\mathcal{E} = \mathcal{E}$. Furthermore, if $B$ is the additive semigroup generated by $\mathcal{E}$, then $B = B$. Let $\mathcal{G}$ denote the set of all subsets $A$ of K such that (i) $A$ is an additive semigroup, and (ii) $\mathcal{E} = A$. Finally,

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for any subset $A$ of $K$, denote by $A^I$ the set of all elements $b$ of $K$ for which there exists an element $a$ in $A$ such that $ba$ is in $A$.

**Lemma 1.** If $A$ is in $E$, then $A^I$ is in $E$ also.

**Proof.** If $b \in A^{18}$, then $b$ is a product of elements

$$a_1, \ldots, a_n, k_1, \ldots, k_m, k_1, \ldots, k_m, \quad a_i \in A^I, \quad k_i \in K^*,$$

in some order. Since each $a_i \in A^I$, there exist $c_i \in A$ such that $a_i c_i \in A$. Let $a = (a_1 c_1) \cdots (a_n c_n)$, an element of $A$. If some $a_i = 0$, then $b = 0$ and $b \in A^I$ immediately. Otherwise, if all $a_i \neq 0$, $ba \in A$ since $ba$ is a product of the elements

$$c_1, \ldots, c_n, a_1, \ldots, a_n, a_1, \ldots, a_n, k_1, \ldots, k_m, k_1, \ldots, k_m,$$

$c_i \in A, a_i, k_i \in K^*$, in some order. Thus $b \in A^I$ and $A^{18} = A^I$.

To prove that $A^I$ is an additive semigroup, let $ai \in A^I$ with $a_i \neq 0$. Then there exist $c_i \in A$ such that $a_i c_i \in A$. Since

$$(a_1 + a_2) c_1 c_2 c_3 = (a_1 c_1) (a_2 c_2) + a_2 c_1 c_2 c_3,$$

and $c_1 (a_2 c_2)$, $(a_1 c_1) (a_2 c_2)$, and $a_2 c_1 c_2$ all are in $A$, we conclude that $a_1 + a_2 \in A^I$. Thus $A^I \subseteq E$, and the lemma is proved.

**Lemma 2.** If $A \in E$, $A \subseteq K^*$, and $A^I = A$, and if, for $d \in K^*$, neither $d$ nor $-d$ is in $A$, then the element $B$ of $E$ generated by $A \cup \{d\}$ also is contained in $K^*$.

**Proof.** If $C = [A \cup \{d\}]^g$, then $B$ is the additive semigroup generated by $C$. If $c \in C$, then either $c \in A$ or $c$ is a product of elements

$$d, a_1, \ldots, a_n, k_1, \ldots, k_m, k_1, \ldots, k_m, \quad a_i \in A, \quad k_i \in K^*,$$

in some order. In this latter case, $dc \in A$. Thus $C = A \cup F$, where $dF \subseteq A$; and $B = A \cup F' \cup (A + F')$, where $F'$ is the additive semigroup generated by $F$. Since $dF \subseteq A$, evidently $F' \subseteq K^*$. To prove that $B \subseteq K^*$, let us assume $0 \in B$. Then $0 = a + f$ for some $a \in A$ and $f \in F'$. Since $(-d)(-f) \in A$, also $(-d)a \in A$. However $A^I = A$, and therefore $-d \in A$. This contradiction proves that $B \subseteq K^*$, and the lemma follows.

**Proof of theorem.** If $K$ is orderable, so that $K^* = P \cup (-P)$ for some additive and multiplicative semigroup $P$, then it is easy to see that $S \subseteq P \subseteq K^*$.

On the other hand, if $S \subsetneq K^*$, let $\mathcal{A}$ be the subset of $E$ containing all $A$ such that $S \subseteq A \subseteq K^*$. By Zorn’s lemma, $\mathcal{A}$ has a maximal element $P$. In view of Lemma 1, we must have $P^I = P$. That $K^* = P \cup (-P)$ is now an immediate consequence of Lemma 2.
In case $K$ is a division ring, the set $S$ coincides with the set generated by the perfect squares as used by Szele. This follows easily from the identity $xyz = (xy)^2(y^{-1})^2y$, $x, y \in K^*$.

If the domain of integrity $K$ is ordered, say $K^* = P \cup (-P)$, then for an extension $L$ of $K$ the ordering of $K$ can be extended to an ordering of $L$ if and only if $T \subset L^*$, where $T$ is the additive semigroup in $L$ generated by $P^*$. The proof of this result is much the same as that of the above theorem. This generalizes Theorem 2 of Szele's paper to a domain of integrity.

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THE ZEROS OF AN ANALYTIC FUNCTION
OF ARBITRARILY RAPID GROWTH

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1. Introduction. It was shown by Poincaré [4], Borel [1], and others that an integral function may be made "to grow" arbitrarily rapidly along the real axis or along other curves extending to infinity. Ketchum [2] has considered the corresponding problem for more general point sets. He investigated sets such that, for any given function $G(z) \geq 0$, there exists a function $f(z)$ which is analytic except where $G(z)$ is unbounded and which satisfies the inequality

$$|f(z)| \leq G(z)$$

for every point $z$ of the set.

In the publication of his results Ketchum [2] proposed a corresponding problem in which the additional restriction is placed on the function $f(z)$ that it be nonvanishing except at certain specified points of the complement of the set. In particular, suppose $S_1, S_2, \cdots$ is an infinite sequence of simply-connected regions whose closures are nonintersecting and whose only "sequential limit point" is the point at infinity. Then, if $\{M_i\}$ is any preassigned sequence of positive constants, does there exist a nonvanishing integral function $f(z)$ such that $|f(z)| \geq M_i$ when $z \in S_i$?

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2 Numbers in brackets refer to the bibliography at the end of the paper.