IN Variant METRICS IN GROUPS (SOLUTION OF A PROBLEM OF BANACH)

V. L. Klee, JR. 1

Introduction. If G is a semi-group and \( \rho \) a metric on G, \( \rho \) will be called left invariant if \( \rho(gx, gy) = \rho(x, y) \) whenever \( \{g, x, y\} \subset G \), right invariant if always \( \rho(xg, yg) = \rho(x, y) \), and invariant if it is both right and left invariant. If T is a topological space and \( \rho \) a metric on T, we shall say that T admits \( \rho \) if the \( \rho \)-topology of T agrees with its original topology. G. Birkhoff [2] and Kakutani [5] proved that a Hausdorff group admits a left invariant metric if and only if it satisfies the first axiom of countability. §1 below contains some remarks on invariant metrics, including a slight sharpening of the theorem just mentioned.

A topological space will be called topologically complete if it admits a metric under which it is complete. The principal result of this note (2.4) is that if G is a Hausdorff group which is abelian, metrizable, and topologically complete, then G admits an invariant metric under which it is complete. As applied to linear metric spaces, this answers affirmatively a question of Banach [1, p. 232].

I am indebted to Professor Kakutani for pointing out an oversight in my original version of this note.

1. Invariant metrics.

(1.1) If G is a group with left invariant metric \( \rho \) and neutral element e, then \( \rho(g^{-1}, e) = \rho(g, e) \) whenever \( g \in G \).

(1.2) Suppose G is a group with left invariant metric \( \rho \). Then the following statements are equivalent: (a) \( \rho \) is right invariant; (b) \( \rho \) is invariant under inversion; (c) \( \rho \) is invariant under every inner automorphism of G.

Proof. (a) implies (b): \( \rho(x^{-1}, y^{-1}) = \rho(x^{-1}x, y^{-1}y) = \rho(e, (x^{-1}y)^{-1}) = \rho(e, x^{-1}y) = \rho(x, y) \).

(b) implies (c): \( \rho(gxg^{-1}, gyg^{-1}) = \rho(xg^{-1}, yg^{-1}) = \rho(gx^{-1}, gy^{-1}) = \rho(x^{-1}, y^{-1}) = \rho(x, y) \).

(c) implies (a): \( \rho(xg, yg) = \rho(gxg^{-1}, gyg^{-1}) = \rho(gx, gy) = \rho(x, y) \).

Presented to the Society, December 28, 1951; received by the editors October 9, 1951 and, in revised form, October 18, 1951.

1 National Research Fellow.

§ Numbers in brackets indicate references at the end of the paper.

§ §1 is closely related to work of van Dantzig [3, §§3-4] and could in part be replaced by references to his paper. However, its present form is better suited to our discussion.
(1.3) Suppose $G$ is a semi-group with metric $\rho$. Then invariance of $\rho$ implies

\[(\$) \quad \rho(ab, xy) \leq \rho(a, x) + \rho(b, y) \quad \text{whenever } \{a, b, x, y\} \subset G.\]

If $G$ is a group, invariance of $\rho$ is equivalent to $(\$)$.

**Proof.** If $\rho$ is invariant, then

$$\rho(ab, xy) \leq \rho(ab, xb) + \rho(xb, xy) = \rho(a, x) + \rho(b, y).$$

If $(\$)$ holds and $G$ is a group, then

$$\rho(gu, gv) \leq \rho(g, g) + \rho(u, v) = \rho(g^{-1}gu, g^{-1}gv) \leq \rho(gu, gv),$$

so $\rho$ is left invariant. A similar argument shows that $\rho$ is right invariant and completes the proof.

(1.4) Suppose $G$ is a semi-group with invariant metric $\rho$, $(G^*, \rho^*)$ is the metric completion of $(G, \rho)$. Let multiplication in $G^*$ be defined by termwise multiplication of Cauchy sequences. Then $G^*$ is a semi-group, $\rho^*$ is invariant on $G^*$, and the natural embedding of $G$ in $G^*$ is an isomorphism as well as an isometry. If $G$ is a group, then so is $G^*$.

**Proof.** As usual, two Cauchy sequences $u_\alpha$ and $v_\alpha$ of $G$ are said to be equivalent ($u_\alpha \sim v_\alpha$) if $\lim \rho(u_\alpha, v_\alpha) = 0$. The elements of $G^*$ are equivalence classes of Cauchy sequences, with $\rho^*([x_\alpha], [y_\alpha]) = \lim \rho(x_\alpha, y_\alpha)$. Now suppose $x_\alpha$, $x'_\alpha$, $y_\alpha$, $y'_\alpha$ are Cauchy sequences with $x_\alpha \sim x'_\alpha$ and $y_\alpha \sim y'_\alpha$. For each $n$ let $z_n = x_n y_n$ and $z'_n = x'_n y'_n$. From $(\$)$ above it follows that $z_n$ and $z'_n$ are Cauchy sequences, with $z_n \sim z'_n$. Thus we can define multiplication in $G^*$ by $[x_\alpha][y_\alpha] = [(x_1 y_1, x_2 y_2, \cdots)]$ and $G^*$ becomes a semi-group in which $G$ is isometrically and isomorphically embedded. It is easy to see that $\rho^*$ is invariant on $G^*$. And if $G$ is a group, the fact that $G^*$ is a group follows from invariance under inversion of $\rho$. The proof is complete.

(1.5) Suppose $G$ is a group having a Hausdorff topology. Then (for $i = 1, 2$) the statements $(a_i)$ and $(b_i)$ below are equivalent:

(a$_1$) $G$ admits a left invariant metric;

(b$_1$) $G$ is first countable at $e$, the group operations are continuous at $e$, and $yg|g \in G$ is continuous for each $y \in G$;

(a$_2$) $G$ admits an invariant metric;

(b$_2$) $G$ is a Hausdorff group which admits at $e$ a countable complete system of neighborhoods, each invariant under every inner automorphism of $G$.

**Proof.** Suppose first that $(a_1)$ holds. Then two of the assertions of $(b_1)$ are obvious. Continuity at $e$ follows for inversion from (1.1) and for multiplication from the inequality,

$$\rho(xy, e) = \rho(y, x^{-1}) \leq \rho(y, e) + \rho(e, x^{-1}) = \rho(y, e) + \rho(x, e).$$
Now suppose \( (a_2) \) holds. That \( G \) admits at \( e \) a system of neighborhoods of the specified type follows from (1.2), as does the fact that inversion is continuous. Continuity of multiplication follows from \((\$)\) of (1.3).

That \( (b_i) \) implies \( (a_i) \) (for \( i = 1, 2 \)) follows without difficulty from a general theorem of Kakutani \([6]\) and also from an examination of the proof of Birkhoff \([2]\). In each case there is a countable complete system \( V_e \) of neighborhoods of \( e \) such that for all \( k \), \( V_k = V_e^{-1} \) and \( V_k \subset V_{k-1} \). Assuming \( (b_2) \), these can be taken invariant under every inner automorphism of \( G \). Then (following Birkhoff) let \( \delta(x, y) = \inf_{\gamma \in V_k} (1/2)^k \) and \( \rho(x, y) = \inf_{z \in \mathbb{G}, n \in \mathbb{N}} \sum_{k=1}^{n} \delta(u_{k-1}, u_k) \). In both cases \( \rho \) is a left invariant metric compatible with the topology of \( G \), and under \( (b_2) \) \( \rho \) is actually invariant.

2. Complete invariant metrics. An argument essentially the same as that in (2.1) and (2.2) below is used by Mazur and Sternbach \([8, \text{p. 50}]\) to prove that a \( G_\alpha \) linear subset of a Banach space must actually be closed.

(2.1) Suppose \( S \) is a second category topological group and \( X \) is a subgroup of \( S \). Then \( S - X \) is either empty or of second category in \( S \).

**Proof.** Suppose \( y \in S - X \). Then \( yX \in S - X \), and if \( S - X \) is of first category, so is \( yX \); but then so is \( X \), and hence \( S \) itself, a contradiction completing the proof.

(2.2) Suppose \( S \) and \( X \) are as in (2.1), and \( X \) is a dense \( G_\delta \) subset of \( S \). Then \( X = S \).

**Proof.** We have \( X = \bigcap_i X_i \), where each \( X_i \) is a dense open set. But then each set \( S - X_i \) is closed and nowhere dense, so \( S - X \) (being the union of these sets) is of first category. The desired conclusion follows from (2.1).

(2.3) Suppose \( G \) is a group with invariant metric \( \rho \). Then if the space \((G, \rho)\) is topologically complete, \( G \) is actually complete under \( \rho \).

**Proof.** Let \((G^*, \rho^*)\) be the metric completion of \((G, \rho)\) and recall the facts stated in (1.4): \((G^*, \rho^*)\) is a topological group in which \((G, \rho)\) is isomorphically and isometrically embedded. Sierpinski has proved \([9]\) that a topologically complete metric space is a \( G_\delta \) set relative to every metric space in which it is topologically embedded. Thus from (2.2) it follows that \( G \) (as embedded in \( G^* \)) is identical with \( G^* \), and hence \( G \) is complete under \( \rho \).

Neither (1.4) nor (2.3) is valid if \( \rho \) is assumed merely to be left-invariant. For let \( G \) be the group of all homeomorphisms of \([0, 1]\) onto itself, with topology supplied by the metric \( d(u, v) = \sup_{x \in [0, 1]} |u(x) - v(x)| \). Then \( G \) is a Hausdorff group and thus admits a left-invariant metric. However, Dieudonné \([4]\) has ob-
served that $G$ cannot be isomorphically embedded in a complete topological group. (I am indebted to Dr. Ernest Michael for this reference.) It is further easy to see that $G$ is a $G_δ$ set in the set of all continuous monotone functions on $[0, 1]$, and that the latter set is complete in the metric $d$. Thus $G$ is topologically complete.

From (2.3) and (1.5) we obtain:

(2.4) Suppose $G$ is a Hausdorff group whose neutral element admits a countable complete system of neighborhoods, each invariant under every inner automorphism of $G$. Then $G$ admits an invariant metric, and if topologically complete must be complete under every invariant metric.

Dr. Michael has pointed out that (2.4) implies:

(2.5) The space of a metric abelian group is topologically complete if and only if it is complete in the uniformity determined by the neighborhoods of the neutral element.

A corollary of (2.4) is:

(2.6) Every complete linear metric space can be metrized as a (complete) space of type (F).

This answers affirmatively a question of Banach [1, p. 232]. The question has also been considered by G. G. Lorentz [7]. His principal results are reduced by (2.6) to previous results of other authors.

Another consequence of (2.4) is:

(2.7) A normed linear space is a Banach space if and only if it is topologically complete.

References


The Institute for Advanced Study