INVARIANT METRICS IN GROUPS (SOLUTION OF A PROBLEM OF BANACH)

V. L. KLEE, JR.[1]

Introduction. If G is a semi-group and ρ a metric on G, ρ will be called *left invariant* if $\rho(gx, gy) = \rho(x, y)$ whenever $\{g, x, y\} \subset G$, *right invariant* if always $\rho(xg, yg) = \rho(x, y)$, and *invariant* if it is both right and left invariant. If T is a topological space and ρ a metric on T, we shall say that T *admits* ρ if the ρ-topology of T agrees with its original topology. G. Birkhoff [2][2] and Kakutani [5] proved that a Hausdorff group admits a left invariant metric if and only if it satisfies the first axiom of countability. §1 below contains some remarks on invariant metrics,[3] including a slight sharpening of the theorem just mentioned.

A topological space will be called *topologically complete* if it admits a metric under which it is complete. The principal result of this note (2.4) is that *if G is a Hausdorff group which is abelian, metrizable, and topologically complete, then G admits an invariant metric under which it is complete.* As applied to linear metric spaces, this answers affirmatively a question of Banach [1, p. 232].

I am indebted to Professor Kakutani for pointing out an oversight in my original version of this note.

1. Invariant metrics.

(1.1) *If G is a group with left invariant metric ρ and neutral element e, then $\rho(g^{-1}, e) = \rho(g, e)$ whenever $g \in G$.*

(1.2) *Suppose G is a group with left invariant metric ρ. Then the following statements are equivalent*: (a) *ρ is right invariant*; (b) *ρ is invariant under inversion*; (c) *ρ is invariant under every inner automorphism of G.*

PROOF. (a) implies (b): $\rho(x^{-1}, y^{-1}) = \rho(x^{-1}x, y^{-1}x) = \rho(e, (x^{-1}y)^{-1}) = \rho(e, x^{-1}y) = \rho(x, y)$.

(b) implies (c): $\rho(gxg^{-1}, gyg^{-1}) = \rho(xg^{-1}, yg^{-1}) = \rho(gx^{-1}, gy^{-1}) = \rho(x^{-1}, y^{-1}) = \rho(x, y)$.

(c) implies (a): $\rho(xg, yg) = \rho(gxgg^{-1}, gygg^{-1}) = \rho(gx, gy) = \rho(x, y)$.

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[1] National Research Fellow.

[2] Numbers in brackets indicate references at the end of the paper.

[3] §1 is closely related to work of van Dantzig [3, §§3–4] and could in part be replaced by references to his paper. However, its present form is better suited to our discussion.

(1.3) *Suppose G is a semi-group with metric ρ. Then invariance of ρ implies*

$$(\$) \qquad \rho(ab, xy) \leqq \rho(a, x) + \rho(b, y) \qquad \textit{whenever } \{a, b, x, y\} \subset G.$$

If G is a group, invariance of ρ is equivalent to (\$).

PROOF. If ρ is invariant, then

$$\rho(ab, xy) \leqq \rho(ab, xb) + \rho(xb, xy) = \rho(a, x) + \rho(b, y).$$

If (\$) holds and G is a group, then

$$\rho(gu, gv) \leqq \rho(g, g) + \rho(u, v) = \rho(u, v) = \rho(g^{-1}gu, g^{-1}gv) \leqq \rho(gu, gv),$$

so ρ is left invariant. A similar argument shows that ρ is right invariant and completes the proof.

(1.4) *Suppose G is a semi-group with invariant metric ρ, (G^*, ρ^*) is the metric completion of (G, ρ). Let multiplication in G^* be defined by termwise multiplication of Cauchy sequences. Then G^* is a semi-group, ρ^* is invariant on G^*, and the natural embedding of G in G^* is an isomorphism as well as an isometry. If G is a group, then so is G^*.*

PROOF. As usual, two Cauchy sequences u_α and v_α of G are said to be equivalent ($u_\alpha \sim v_\alpha$) if $\lim_i \rho(u_i, v_i) = 0$. The elements of G^* are equivalence classes of Cauchy sequences, with $\rho^*([x_\alpha], [y_\alpha]) = \lim_i \rho(x_i, y_i)$. Now suppose x_α, x'_α, y_α, y'_α are Cauchy sequences with $x_\alpha \sim x'_\alpha$ and $y_\alpha \sim y'_\alpha$. For each n let $z_n = x_n y_n$ and $z'_n = x'_n y'_n$. From (\$) above it follows that z_α and z'_α are Cauchy sequences, with $z_\alpha \sim z'_\alpha$. Thus we can define multiplication in G^* by $[x_\alpha][y_\alpha] = [(x_1y_1, x_2y_2, \cdots)]$ and G^* becomes a semi-group in which G is isometrically and isomorphically embedded. It is easy to see that ρ^* is invariant on G^*. And if G is a group, the fact that G^* is a group follows from invariance under inversion of ρ. The proof is complete.

(1.5) *Suppose G is a group having a Hausdorff topology. Then* (*for* $i = 1, 2$) *the statements* (a_i) *and* (b_i) *below are equivalent*:

(a_1) *G admits a left invariant metric*;

(b_1) *G is first countable at e, the group operations are continuous at e, and $yg | g \in G$ is continuous for each $y \in G$*;

(a_2) *G admits an invariant metric*;

(b_2) *G is a Hausdorff group which admits at e a countable complete system of neighborhoods, each invariant under every inner automorphism of G.*

PROOF. Suppose first that (a_1) holds. Then two of the assertions of (b_1) are obvious. Continuity at e follows for inversion from (1.1) and for multiplication from the inequality,

$$\rho(xy, e) = \rho(y, x^{-1}) \leqq \rho(y, e) + \rho(e, x^{-1}) = \rho(y, e) + \rho(x, e).$$

Now suppose (a_2) holds. That G admits at e a system of neighborhoods of the specified type follows from (1.2), as does the fact that inversion is continuous. Continuity of multiplication follows from ($) of (1.3).

That (b_i) implies (a_i) (for $i=1, 2$) follows without difficulty from a general theorem of Kakutani [6] and also from an examination of the proof of Birkhoff [2]. In each case there is a countable complete system V_α of neighborhoods of e such that for all k, $V_k = V_k^{-1}$ and $V_k^3 \subset V_{k-1}$. Assuming (b_2), these can be taken invariant under every inner automorphism of G. Then (following Birkhoff) let $\delta(x, y) = \inf_{x^{-1}y \in V_k} (1/2)^k$ and $\rho(x, y) = \inf_{x=u_0, u_n=y} \sum_{k=1}^n \delta(u_{k-1}, u_k)$. In both cases ρ is a left invariant metric compatible with the topology of G, and under (b_2) ρ is actually invariant.

2. **Complete invariant metrics.** An argument essentially the same as that in (2.1) and (2.2) below is used by Mazur and Sternbach [8, p. 50] to prove that a G_δ linear subset of a Banach space must actually be closed.

(2.1) *Suppose S is a second category topological group and X is a subgroup of S. Then $S-X$ is either empty or of second category in S.*

Proof. Suppose $y \in S-X$. Then $yX \subset S-X$, and if $S-X$ is of first category, so is yX; but then so is X, and hence S itself, a contradiction completing the proof.

(2.2) *Suppose S and X are as in* (2.1), *and X is a dense G_δ subset of S. Then $X=S$.*

Proof. We have $X = \bigcap_i X_i$, where each X_n is a dense open set. But then each set $S-X_n$ is closed and nowhere dense, so $S-X$ (being the union of these sets) is of first category. The desired conclusion follows from (2.1).

(2.3) *Suppose G is a group with invariant metric ρ. Then if the space (G, ρ) is topologically complete, G is actually complete under ρ.*

Proof. Let (G^*, ρ^*) be the metric completion of (G, ρ) and recall the facts stated in (1.4): (G^*, ρ^*) is a topological group in which (G, ρ) is isomorphically and isometrically embedded. Sierpinski has proved [9] that a topologically complete metric space is a G_δ set relative to every metric space in which it is topologically embedded. Thus from (2.2) it follows that G (as embedded in G^*) is identical with G^*, and hence G is complete under ρ.

Neither (1.4) nor (2.3) is valid if ρ is assumed merely to be left-invariant. For let G be the group of all homeomorphisms of $[0, 1]$ onto itself, with topology supplied by the metric $d(u, v) = \sup_{x \in [0,1]} |u(x) - v(x)|$. Then G is a Hausdorff group and thus admits a left-invariant metric. However, Dieudonné [4] has observed that G cannot be isomorphically embedded in a complete topological group. (I am indebted to Dr. Ernest Michael for this reference.) It is further easy to see that G is a G_δ set in the set of all continuous monotone functions on $[0, 1]$, and that the latter set is complete in the metric d. Thus G is topologically complete.

From (2.3) and (1.5) we obtain:

(2.4) *Suppose G is a Hausdorff group whose neutral element admits a countable complete system of neighborhoods, each invariant under every inner automorphism of G. Then G admits an invariant metric, and if topologically complete must be complete under every invariant metric.*

Dr. Michael has pointed out that (2.4) implies:

(2.5) *The space of a metric abelian group is topologically complete if and only if it is complete in the uniformity determined by the neighborhoods of the neutral element.*

A corollary of (2.4) is:

(2.6) *Every complete linear metric space can be metrized as a (complete) space of type* (F).

This answers affirmatively a question of Banach [1, p. 232]. The question has also been considered by G. G. Lorentz [7]. His principal results are reduced by (2.6) to previous results of other authors.

Another consequence of (2.4) is:

(2.7) *A normed linear space is a Banach space if and only if it is topologically complete.*

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