NOTE ON OVERCONVERGENCE IN SEQUENCES OF
ANALYTIC FUNCTIONS

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In a recent paper [1] the authors have set forth measures of the
degree of convergence of families of analytic functions of best
approximation (or of minimum norm) in the sense of least \( p \)th powers.
It is possible to study overconvergence of certain sequences of these
functions by employing the concept of exact harmonic majorant de-
veloped by Walsh [2]. It is the purpose of the present note to indi-
cate this relationship and some of its consequences.

Let \( R \) be a finite region bounded by a finite sum \( C_1 \) of mutually dis-
joint Jordan curves. Let \( S \) be a closed set interior to \( R \), bounded by a
finite sum \( C_0 \) of mutually disjoint Jordan curves, but such \( C_0 \) sepa-
rating no point of \( R - S \) from \( C_1 \). Let \( \phi(z) \) be the function harmonic
in \( R - S \), continuous in the closure of \( R - S \), equal to zero and unity
on \( C_0 \) and \( C_1 \), respectively. Let \( C_\sigma, 0 < \sigma < 1 \), denote the locus \( \phi(z) = \sigma \),
while \( R_\sigma \) is the region consisting of \( S \) plus the points of \( R - S \) where
\( \phi(z) < \sigma \). Suppose that \( f(z) \) is analytic throughout \( R, 0 < p < 1 \), but
coincides on \( S \) with no function analytic throughout \( R_\rho \), for any \( \rho' > \rho \).

In the class of functions \( F_M(z) \) analytic throughout \( R \) and such that
the integral mean

\[
\mu_\tau(F_M, C_1) = \left\{ \frac{1}{\tau} \int_{C_1} |F_M(z)|^\tau d\psi \right\}^{1/\tau} \leq M
\]

\( (\psi(z) \) conjugate to \( \phi(z) \) in \( R - S \), \( \tau = \int_{C_1} d\psi = -\int_{C_0} d\psi \) ), the function
\( F_M(z) \) is defined as the (or a) function of best approximation to \( f(z) \) on
\( S \) if \( \mu_\rho(f - F_M, C_0) \) is least. For certain sequences of values of \( M \), in-
cluding sequences \{ \( M_\alpha \) \} for which \( \log M_\alpha/n \to a, 0 < a < \infty \), it was
shown [1] that

\[
\limsup_{n \to \infty} \mu_t(f - F_{M_\alpha}, C_\sigma)^{1/n} = e^{(\sigma - \rho)/(1-\rho)},
\]

provided \( 0 \leq \sigma < \rho \), for all \( t, 0 < t \leq \infty \); moreover, for \( 0 < t \leq \infty \) when
\( \rho \leq \sigma < 1 \) and for \( 0 < t \leq q \) when \( \sigma = 1 \),

\[
\limsup_{n \to \infty} \mu_t(F_{M_\alpha}, C_\sigma)^{1/n} = e^{(\sigma - \rho)/(1-\rho)}.
\]

For \( t = \infty \), equations (1) and (2) become \( (f_n(z) \equiv F_{M_\alpha}(z)) \)

\[
\limsup_{n \to \infty} \left[ \max_{z \in C_\sigma} |f(z) - f_n(z)| \right]^{1/n} = e^{(\sigma - \rho)/(1-\rho)},
\]

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0 \leq \sigma < \rho$, while

\[
\limsup_{n \to \infty} \left[ \max_{z \in C_{\sigma}} |f_n(z)|, \ z \in C_{\sigma} \right]^{1/n} = e^{\sigma (\sigma - \rho) / (1 - \rho)},
\]

\[
\rho \leq \sigma < 1. \text{ Consequently,}
\]

\[
\limsup_{n \to \infty} \left[ \max_{z \in C_{\sigma}} |f_{n+1}(z) - f_n(z)|, \ z \in C_{\sigma} \right]^{1/n} = e^{\sigma (\sigma - \rho) / (1 - \rho)},
\]

0 \leq \sigma < 1. (That the left-hand member of (5) is not greater than the right-hand member is immediate from (3) and (4). But unless the equality holds, the sequence \{f_n(z)\} converges uniformly throughout some \( \bar{R}_n \), \( \rho' > \rho \), to a function coinciding on \( S \) with \( f(z) \). This is contrary to the assumption on \( f(z) \).

Equation (5) constitutes a necessary and sufficient condition [2, Corollary 2 of Theorem 4] that the function \( V(z) = a(\phi(z) - \rho) / (1 - \rho) \) be an exact harmonic majorant of the sequence \[f_{n+1}(z) - f_n(z)]^{1/n} in each \( R_\sigma \), \( 0 \leq \sigma < 1 \), and hence in \( R \). Thus the conditions of [2, Theorems 5 and 6] are satisfied. We can conclude, therefore, that if for some sequence \{n_k\} and some \( \sigma \) the left-hand member of (3) or (4) is less than the right-hand member, then \[f_{n_k}(z)\} converges throughout some neighborhood of each point of \( C_{\sigma} \) where \( f(z) \) is analytic, and conversely. This result is, of course, much broader than our previous result [1, Theorem 4]. Further, if \{f_{n_k}(z)\} exhibits overconvergence on two disjoint arcs of \( C_{\sigma} \) having a common end point \( \alpha \), then \( z = \alpha \) cannot be an isolated singularity of \( f(z) \).

This reasoning and these conclusions remain valid if \{f_n(z)\} is no longer a sequence of functions of best approximation, but any sequence of functions, each analytic throughout \( R \) with integral mean \( \mu_\sigma(f_n, C_\sigma) \) not greater than \( M_\sigma \) (log \( M_\sigma / n \to \alpha \)) and (1) and (2) valid with \( t = q \) for \( \sigma = 0 \) and \( \sigma = 1 \), respectively. Analogous remarks can be made for the functions of minimum norm: among the functions \( G_m(z) \) analytic in \( R \) with \( \mu_\sigma(f - G_m, C_\sigma) \leq m \), the function \( G_m(z) \) is the (or a) function for which \( \mu_\sigma(G_m, C_\sigma) \) is least. There are many other consequences [2] involving rapidly convergent sequences, zeros of approximating functions, etc., which follow immediately.

**References**


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