In this paper we consider postulates expressed in terms of "segments," "medians," and "betweenness." Characterizations are obtained for trees, lattices, and partially ordered sets. In general a characterization is given by a system of three postulates. These systems fall in pairs; systems of a pair have two postulates in common. An algebra which has both lattices and trees as special cases is given in the final section.

1. Segments. Consider a set $S$ of elements $a, b, c, \ldots$ such that to each pair $a, b$ of elements in $S$ there corresponds a unique subset of $S$ denoted by $(a, b)$ and called the segment from $a$ to $b$. By assumption, these segments have as properties:

(S) To each set of three elements $a, b, c$ there corresponds an element $d$ such that

\[(a, b) \cap (b, c) = (d, b, c).\]

(T) $(a, b) \subset (a, c)$ implies $(a, b) \cap (b, c) = \{b\}$.

The segments of Duthie [1] are segments in this sense (see §3). Paths in a tree are also segments in this sense (see §2).

Setting $a = b = c$ in (T) we have

\[\{a\} = (a, a).\]

From this, and from (T) with $b = c$, we have

\[b \in (a, b),\]

\[a \in (a, b).\]

**Proof.** From (S), we may choose $d$ so that $(a, a) \cap (a, b) = (a, d)$.

By (1.1) and (1.2), $d \in (a, a) \subset (a, a) = \{a\}$. Hence $d = a$ and $a \in (a, d) \subset (a, b)$.

\[b \in (a, b) \quad \text{if and only if} \quad (a, b) \subset (a, c).\]

Received by the editors June 12, 1950 and, in revised form, September 20, 1951.

1 This paper was written while the author was under contract to the Office of Naval Research.

* Numbers in brackets refer to the bibliography at the end of the paper.

369
Proof. The sufficiency proof is trivial. To prove necessity choose \(d\) so that \((b, a) \cap (a, c) = (a, d)\). Clearly \(b \in (a, d)\). From (1.4) and (T), 
\((a, d) \cap (d, b) = \{d\}\). Hence \(b \in \{d\}\), \(b = d\), and \((a, b) = (a, d) \subseteq (a, c)\).

From (1.5) and (T) we have

\[
(1.6) \quad b \in (a, c) \implies (a, b) \cap (b, c) = \{b\}.
\]

\[
(1.7) \quad b \in (a, c) \text{ and } c \in (a, b) \implies b = c.
\]

Thus if \((a, b) = (a, c)\) we have \(b = c\).

Proof. By (1.5), \((a, b) \subseteq (a, c) \subseteq (a, b)\). Hence these segments are equal. From (1.4) and (1.6), \(\{b\} = (a, b) \cap (b, c) = (a, c) \cap (c, b) = \{c\}\).

As a corollary of (1.7) we have

\[
(1.8) \quad \text{The element } d \text{ of } (S) \text{ is unique.}
\]

\[
(1.9) \quad b \in (a, c) \text{ if and only if } (a, b) \cup (b, c) \subseteq (a, c).
\]

Proof. The condition is necessary by (1.4) and (1.5). It is sufficient by (1.2) and (1.3).

\[
(1.10) \quad (a, b) \cap (b, c) = (b, d) \implies (a, d) \cap (d, c) = \{d\}.
\]

Proof. We note \(d \in (a, b)\) and \(d \in (b, c)\). From (1.9), \((a, b) \supseteq (a, d) \cup (d, b)\) and \((b, c) \supseteq (b, d) \cup (d, c)\). Taking set intersections we obtain \((b, d) \supseteq (b, d) \cap [(a, d) \cap (d, c)]\). From this and from (1.6), \((a, d) \cap (d, c) = (a, d) \cap (d, c) \cap (b, d) = \{d\} \cap (d, c) = \{d\}\).

2. Tree segments. The word tree is probably most often used in mathematics to denote a finite connected linear graph which contains no cycles [2, p. 47]. However a connected acyclic union of closed Jordan arcs is sometimes called a tree and in lattice theory [3, p. 47] we find the word used in still a third sense. Seeking characteristics common to the several types of objects known as trees we arrive at the definition below. (Birkhoff’s trees can be imbedded in our trees. Trees in our sense which are finite are trees in König’s sense.)

A tree is defined as a set of elements which satisfies (S), (T), and:

\((U_1)\) \((a, b) \cap (b, c) = \{b\}\) implies \((a, b) \cup (b, c) = (a, c)\).

From this definition, (1.9), and (1.6) we have

\[
(2.1) \quad b \in (a, c), \quad (a, b) \cap (b, c) = \{b\}, \quad \text{and } (a, b) \cup (b, c) = (a, c) \text{ are equivalent conditions.}
\]

\[
(2.2) \quad (a, b) \cap (b, c) = (b, d) \text{ if and only if } \{d\} = (a, b) \cap (b, c) \cap (c, a).
\]

Hence the last term always represents a set containing a single point.

Proof. If \((a, b) \cap (b, c) = (b, d)\), we have by (1.10) that \((a, d) \cap (d, c) = \{d\}\). Using (2.1) and the distributive law, we have \((a, b)\).
\( \cap (b, c) \cap (c, a) = (b, d) \cap (c, a) = (b, d) \cap [(a, d) \cup (d, c)] = \{d\} \cup \{d\} \)

The sufficiency of the condition follows from \( (S) \) and the necessity of the condition.

**Definition.** The unique element given by \( (a, b) \cap (b, c) \cap (c, a) \)
is called the median of \( a, b, \) and \( c \) and is denoted by \( (a, b, c) \).
Properties of this ternary operation are derived in \( \S 5 \).

(2.3) For all \( a, b, \) and \( c \), \( (a, b) \cap (a, c) \cup (c, b) \).

**Proof.** Let \( d = (a, b, c) \). From (2.1), \( (a, b) = (a, d)^\cup (d, b) \). From (1.5), \( (a, d) \cup (a, c) \) and \( (d, b) \cup (c, b) \).

**Definition.** We say \( b \) is between \( a \) and \( c \) and write \( abc \) if and only if \( b \in (a, c) \).

As a consequence of (1.1) we have

(2.4) \( aba \) if and only if \( a = b \).

As a consequence of (2.2) we have

(2.5) To elements \( a, b, \) and \( c \) there corresponds a unique element \( d \)
such that \( adb, bdc, \) and \( cda \).

(2.6) If we have both \( abc \) and \( bde \), then we have either \( ebd \) or \( eba \), perhaps both.

**Proof.** Let \( b \in (a, c) \) and \( d \in (b, e) \). By (2.3), if \( b \in (a, e) \) then \( b \in (e, c) \). Again, if \( b \in (c, d) \) then \( b \in (d, e) \). But from (1.7) we have in this case the contradiction \( b = d \in (c, d) \).

3. **Lattice segments.** Consider a set \( S \) satisfying \( (S), (T), \) and:

(U2) There are elements 0 and 1 in \( S \) such that if \( (0, r) \cap (0, s) \subset (0, a) \cap (0, b) \) and \( (I, r) \cap (I, s) \subset (I, a) \cap (I, b) \), then \( (a, b) \subset (r, s) \).

(3.1) \( r \in (0, a) \) if and only if \( a \in (r, I) \).

**Proof.** Let \( b = s = I \) in (U2) to prove necessity. The proof of sufficiency is dual.

Letting \( r = 0 \) in (3.1), we have

(3.2) For all \( a, a \in (0, I) \).

(3.3) If \( (0, a) \cap (0, b) = (0, r) \) and \( (I, a) \cap (I, b) = (I, s) \) then \( (a, b) = (r, s) \).

**Proof.** From \( s \in (I, a) \cap (I, b) \) and from (3.1) we have that \( a \) and \( b \in (0, s) \). Hence \( (0, r) = (0, a) \cap (0, b) \subset (0, s) \). By (3.1), \( (I, s) \subset (I, r) \).

A double application of (U2) gives \( (a, b) \subset (r, s) \) and \( (r, s) \subset (a, b) \).

**Definition.** We write \( r = ab \) if and only if \( (0, r) = (0, a) \cap (0, b) \).
We write \( s = a + b \) if and only if \( (I, s) = (I, a) \cap (I, b) \).

(3.4) \( S \) is a lattice with bounds 0 and \( I \).

**Proof.** The commutative, associative, and idempotent laws are easily derived. Properties of 0 and \( I \) follow easily from (3.2). In proving an absorption law, we let \( a + b = c \) and \( ac = d \). From \( (I, c) = (I, a) \cap (I, b) \) and (3.1), we have \( a \) in \( (0, c) \). Hence \( (0, a) = (0, a) \cap (0, c) = (0, d) \). By (1.7), \( a = d \). The other absorption law follows dually.

(3.5) \( x \in (a, b) \) if and only if \( ab \leq x \leq a + b \).

**Proof.** By (3.3), \( x \in (r, s) \) where \( r = ab \) and \( s = a + b \). As in the proof of (3.3), \( (0, r) \subset (0, s) \). By (1.5), \( (r, s) \subset (0, s) \). By (3.1), \( s \subset (I, x) \). Hence \( (I, s) \subset (I, x) \) and \( x + s = s \). Dually, \( xr = r \). To prove the inequality sufficient, we note \( xr = r \) implies \( (0, r) \subset (0, x) \). Similarly, \( (I, s) \subset (I, x) \). Hence \( (0, a) \cap (0, b) \subset (0, a) \cap (0, x) \) and \( (I, a) \cap (I, b) \subset (I, a) \cap (I, x) \). By (U₂), \( (a, x) \subset (a, b) \) and \( x \in (a, b) \).

(3.6) Postulates S, T, and U₂ characterize distributive lattices with 0 and \( I \).

**Proof.** That these postulates give a distributive lattice with 0 and \( I \) follows from (3.4), (3.5), (1.6), and a theorem of Duthie [1]. Conversely, if in such a distributive lattice we define \( (a, b) \) as the set of all \( x \) such that \( ab \leq x \leq a + b \), the three postulates are easily derived.

4. Betweenness. We now consider a set \( S \) of elements \( a, b, c, \ldots \) in which, for each ordered set of three elements \( a, b, \) and \( c \), there holds or fails to hold a relation denoted by \( abc \) and read “\( b \) is between \( a \) and \( c \).” This relation satisfies the following postulates:

(B) \( aba \rightarrow a = b \).

(C) \( abc \cdot bde \rightarrow (cbd \text{ or } eba) \).

For the interpretation of the notation in (C) see (2.6). We proceed to derive consequences of these two postulates.

(4.1) \( aab \rightarrow baa \).

**Proof.** By (C), \( aab \rightarrow aab \cdot aab \rightarrow (baa \text{ or } baa) \rightarrow baa \).

(4.2) \( abc \rightarrow aab \).

**Proof.** If \( a = c \), \( abc \rightarrow aab \rightarrow a = b \). But by (B) we have \( aaa \), and now \( aaa \rightarrow aab \). If \( a \neq c \), from \( aaa \cdot abc \rightarrow (aab \text{ or } caa) \), we have either \( aab \) or \( caa \cdot abc \rightarrow (aab \text{ or } cac) \rightarrow aab \).

(4.3) \( abc \rightarrow cba \).
Proof. If \( a = b \), this follows from (4.1). If \( a \neq b \), we note by (4.2) and (4.1) that \( abc \rightarrow baa \) and we then have that \( abc \cdot baa \rightarrow (cba \text{ or } aba) \rightarrow cba \).

(4.4) Each of the relations \( aab, abb, bba, \) and \( baa \) implies the other three.

Proof. By (4.1), \( aab \rightarrow baa \) and \( bba \rightarrow abb \). By (4.2), \( baa \rightarrow bba \) and \( abb \rightarrow aab \).

Definition. We say \( a \) is comparable with \( b \) and write \( ab \) if and only if \( aab \) holds. It follows that \( aa \) holds for all \( a \) in \( S \), that \( ab \rightarrow baa \), and that \( ab \) is equivalent to each of the betweenness relations of (4.4). It is unlikely that this notation will be confused with the product notation of §3.

(4.5) \( abc \rightarrow ab \cdot bc \cdot ca \).

Proof. By (4.2) and (4.3), \( abc \rightarrow ab \) and \( abc \rightarrow cba \rightarrow cb \). If \( a = b \), \( cb \rightarrow ca \). If \( a \neq b \), by (4.2) and (C), we have \( abc \rightarrow aab \cdot abc \rightarrow (bab \text{ or } caa) \rightarrow caa \rightarrow ca \).

Definition. We say \( a_1, a_2, \ldots, a_n \) form a chain and denote this by \( a_1a_2 \cdots a_n \) if and only if \( a_ia_a \) holds for \( 1 \leq i \leq j \leq k \leq n \). We note the definition is consistent with our previous notation when \( n = 2, 3 \).

Clearly, \( a_1a_2 \cdots a_n \) implies both \( a_na_{n-1} \cdots a_1 \) and \( a_ia_{i+1} \cdots a_j \) for \( 1 \leq i \leq j \leq n \). Moreover when \( a_i = a_j \), we have \( a_i = a_{i+1} = \cdots = a_{j-1} = a_j \).

(4.6) \( abc \cdot bcd \cdot b \neq c \rightarrow abcd \).

Proof. When \( b \neq c \) we have \( abc \cdot bcd \rightarrow (abc \text{ or } dba) \rightarrow ab \). Similarly, \( dcb \cdot cba \rightarrow acd \).

(4.7) \( abc \cdot acd \rightarrow abcd \).

Proof. If \( a = c \), \( abc \rightarrow aba \rightarrow a = b \) and the implication holds. If \( a \neq c \), \( acd \cdot cba \rightarrow (dcb \text{ or } aca) \rightarrow dcb \). If \( b = c \), \( acd \rightarrow abd \). Finally we have \( abd \) when \( b \neq c \) by (4.6).

An easy induction proof establishes the following generalization of (4.6) and (4.7).

(4.8) \[ a_1a_2 \cdots a_n \cdot a_1a_nb \rightarrow a_1a_2 \cdots a_nb, \]
and for \( 1 \leq i \leq n-1 \),

\[ a_1a_2 \cdots a_n \cdot a_ia_{i+1} \rightarrow a_1a_2 \cdots a_{i}a_{i+1} \cdots a_n. \]

Finally, we may easily prove
(4.9) \[ abc \cdot acb \rightarrow b = c. \]

(4.10) \[ abc \cdot bd \rightarrow (abd \text{ or } cbd). \]

It may be noted that we have made no attempt to use the intensive survey of betweenness made by Pitcher and Smiley [4] because of their initial assumption that every pair of elements is comparable. (This follows from their Postulate \( \beta \).) It is interesting, however, that (B) and (C) imply, in addition to Postulate \( \alpha \) and (1) and (2), 40 of the 43 transitivities given in Part I of their paper.\(^3\) The three transitivities not implied are their \( T_4, T_7, \) and \( T_{10} \). Transitivity \( T_{10} \) holds, however, in order betweenness (see (6.1)), and \( T_4 \) and \( T_7 \) hold in tree betweenness (see (8.10) and (8.11)). That \( T_4 \) and \( T_7 \) do not hold in order betweenness (without the restriction \( a \neq b \)) exposes a minor error in a comment by Pitcher and Smiley [4, footnote 4].

5. Tree betweenness. Consider a set \( S \) which satisfies (B), (C), and:

(\( D_1 \)) Given \( a, b, \) and \( c \), there exists an \( x \) such that \( axb \cdot bxc \cdot cxa \).

We show in (5.8) that \( S \) is a tree.

First, from (\( D_1 \)) and (4.5), it follows that every pair of elements is comparable.

(5.1) \[ \text{For } a, b \text{ in } S, ab \text{ holds.} \]

(5.2) \[ axb \cdot ayb \rightarrow (axy \text{ or } bxy). \]

**Proof.** From (5.1), \( xy \) holds. From (4.10), \( axb \cdot xy \rightarrow (axy \text{ or } bxy) \). Assume, say, \( axy \) holds. By (4.7), \( axy \) holds.

(5.3) The element \( x \) in (\( D_1 \)) is unique.

**Proof.** Assume \( y \) has the same property. From \( axb \cdot ayb \) and (5.2) we have, say, \( axy \) and hence \( axy \cdot xyb \). By (4.7), we have \( axyc \cdot cxyb \). From \( xyc, cxy, \) and (4.9), \( x = y \).

**Definitions.** The element \( x \) of (\( D_1 \)) is called the median of \( a, b, \) and \( c \) and is denoted by \( (a, b, c) \). The set of all \( x \) such that \( axb \) holds is called the segment from \( a \) to \( b \) and is denoted by \( (a, b) \).

(5.4) \( (a, b) \subseteq (a, c) \) implies \( (a, b) \cap (b, c) = \{b\} \).

**Proof.** We have given \( axb \rightarrow axc \). We are to show \( y = b \) if and only if \( ayb \cdot byc \). Necessity of the condition follows from (5.1). Conversely, noting \( ab \) holds and using (4.8), we have \( abb \rightarrow abc \rightarrow aybyc \rightarrow y = b \).

(5.5) \( (a, b) \cap (b, c) = \{b\} \) implies \( b \in (a, c) \).

\(^3\) The stronger pair of postulates, (C) and their \( \theta \), imply 42 of the 43 transitivities, all except \( T_{10} \).
Proof. We are to show that if $axb \cdot bxc \rightarrow x = b$, then $abc$ holds. This follows by choosing $x = (a, b, c)$.

(5.6) $(a, b) \cap (b, c) = \{b\}$ implies $(a, b) \cup (b, c) = (a, c)$.

Proof. Assume $axb \cdot bxc \rightarrow x = b$. To show $ayc$ holds if and only if we have either $ayb$ or $byc$. By (5.5), $abc$ holds. The conclusion follows from (4.7) and (5.2).

(5.7) If $d = (a, b, c)$, then $(a, b) \cap (b, c) = (b, d)$.

Proof. To show $bxd$ holds if and only if $axb \cdot bxc$. Necessity of the condition is easily seen. Conversely, from (5.2),

$$axb \rightarrow (axdb \text{ or } adxb) \rightarrow (axd \text{ or } dxb),$$

$$bxc \rightarrow (bdxc \text{ or } bdxc) \rightarrow (bxd \text{ or } dxc).$$

If $bxd$ does not hold, we have from (4.8) the contradiction $adcb \cdot xdbc \rightarrow axdxc \rightarrow x = d$.

(5.8) Trees are characterized as sets $S$ satisfying Postulates B, C, and D1.

Proof. This follows from (2.4), (2.5), (2.6), (5.4), (5.6), and (5.7). For later use we derive properties of the median. As a consequence of its definition, $(a, b, c)$ is invariant under cyclic permutations of $a, b,$ and $c$. Hence from (4.3), we have the following.

(5.9) $(a, b, c)$ is invariant under all permutations of $a, b,$ and $c$.

As a consequence of (4.4) and (5.1) we have

(5.10) $(a, x, b) = x$ if and only if $axb$ holds. Thus $(a, a, b) = a$.

(5.11) $((x, a, b), c, x) = (x, a, c)$ or $(x, b, c)$.

Proof. Let $y = (x, a, b)$ and $z = (y, c, x)$. By (4.7), $xzy \cdot yxa \rightarrow xzya \rightarrow xza$. Similarly, we have $xzb$. Since $xzc$ holds, it remains to show that either $asc$ or $bzc$ holds. If $y = z$, from (5.1) and (4.10) we have $axb \cdot zc \rightarrow (asc \text{ or } bzc)$. If $y \neq z$, we note $xzya \rightarrow ayz$ and from (4.6) obtain $azy \cdot yzc \rightarrow aycz \rightarrow azc$.

(5.12) $((a, b, c), (a, b, d), e) = ((c, d, e), a, b)$.

Proof. Let $x = (a, b, c)$, $y = (a, b, d)$, $r = (x, y, e)$, $z = (c, d, e)$, and $s = (a, b, z)$. We are to prove $r = s$. From $axb$, $ayb$, and (5.2), we have $axyb$ or $ayxb$. Since these cases are handled similarly we assume:

(*) $axyb$ holds.
Case I. \( x \not= r \) and \( r \not= y \).

We have from \( xry \) that \( x \not= y \). Using (*), we have

\[
\begin{align*}
byx \cdot bxc & \rightarrow byxc \rightarrow cxy, \\
axy \cdot ayd & \rightarrow axyd \rightarrow xyd, \\
cxy \cdot xyd \cdot xry & \rightarrow cxryd \rightarrow crd, \\
cxryd \cdot ery & \rightarrow eryd \rightarrow erd,
\end{align*}
\]

and

\[
cxryd \cdot erx \rightarrow cxre \rightarrow cre.
\]

But \( crd \cdot dre \cdot ere \) implies \( r = (c, d, e) = s \). Moreover, from (*) and \( xry \), we have \( arb \). Hence from (5.10) we have \( r = (a, r, b) = (a, z, b) = s \).

Case II. \( x \not= r, r = y \).

We omit the proof since it is similar to the proof of Case III.

Case III. \( x = r, r \not= y \).

We may assume \( z \not= x \) for otherwise from \( axb \) we have \( r = x = (a, b, x) = s \). If \( czx \) holds we have \( czx \cdot cxb \rightarrow zxb \) and, similarly, \( zxa \). Then \( axb \cdot bxz \cdot zxa \) imply \( r = x = (a, b, z) = s \). It remains to show that to assume \( czx \) does not hold leads to a contradiction. Thus

\[
\begin{align*}

dzd \cdot zx & \rightarrow (cuz or dzx) \rightarrow dzx, \\
cze \cdot ezx & \rightarrow (cuz or exz) \rightarrow ezx, \\
bez \cdot exy & \rightarrow exzy \rightarrow zxy,
\end{align*}
\]

and

\[
dzx \cdot exy \cdot z \not= x \rightarrow dzx \rightarrow dxy.
\]

Finally, using (*), \( axy \cdot ayd \rightarrow xyd \). From \( dxy \cdot xyd \cdot x \not= y \rightarrow dxyd \) we have the contradiction \( x = d = y \).

Case IV. \( x = r = y \).

To show \( x = (a, b, c) = (a, b, d) \) and \( z = (c, d, e) \) imply \( x = (a, b, z) \).

We note \( cxd \cdot zx \rightarrow (cuz or dzx) \). The cases are similarly treated. Assume, say, that \( czx \) holds. Then \( czx \cdot cxa \rightarrow zxa \) and \( czx \cdot cxb \rightarrow zxb \). Since \( axb \) holds, \( r = x = (a, b, z) = s \).

6. Order betweenness. Consider a set \( S \) satisfying Postulates B, C, and:

\((D2)\) For odd \( n \geq 3 \), \( a_1a_2 \cdot a_3a_4 \cdots a_{n-1}a_2 \cdot a_n \cdot a_1 \) implies either \( a_{n-i}a_i \cdot a_{n-i}a_{i+1} \cdot a_{i+1}a_{i+2} \) for some \( i \), \( 1 \leq i \leq n-2 \).

As a typical application of \((D2)\) we sketch the proof of transitivity \( T_{10} \) of Pitcher and Smiley [4].
(6.1) \( abc \cdot abd \cdot xbc \cdot a \neq b \cdot b \neq c \rightarrow xbd \).

**Proof.** The hypotheses imply \( cx \cdot xb \cdot bd \cdot da \cdot ac \). By (D₂), we have \( cxb, xbd, bda, dac, \) or \( acx \). If \( cxb \) or \( bda \) holds, then \( b \) equals \( x \) or \( d \) and \( xbd \) holds. If \( dac \) or \( acx \) holds, we easily derive as a contradiction that \( b \) equals \( a \) or \( c \).

In view of (B), (4.3), (4.2), (4.6), and (D₂), it is clear that the following theorem is a corollary of Altwegg’s results [5, Sect. 2].

(6.2) Postulates B, C, and D₂ characterize partially ordered sets to within dual orderings of their connected subsets. Here \( abc \) is equivalent to either \( a \leq b \leq c \) or \( c \leq b \leq a \).

It is possible to characterize a lattice in terms of betweenness by adding a fourth postulate to (B), (C), and (D₂). Then \( abc \) holds if and only if either \( a+b=bc \) or \( ab=b+c \). This is, of course, not the betweenness ordinarily called lattice betweenness. The latter, for distributive lattices, has \( abc \) equivalent to the condition \( ac+b = b(a+c) \).

7. **Chain betweenness.** A chain in the sense used here is often called a completely ordered set or a linearly ordered set. An extensive literature is devoted to discussion of chain betweenness.

Consider the postulate:

(D₃) For \( a \), \( b \), and \( c \) in \( S \) either \( abc \), \( bca \), or \( cab \) holds.

(7.1) Chains are characterized by Postulates B, C, and D₃ or by Postulates B, C, D₁, and D₂.

**Proof.** The first statement follows from a result of Altwegg [5, Sect. 4] and from (D₃), (4.3), (B), (4.6), and (4.7). Since a chain is clearly both a tree and a partially ordered set, it remains to be proved only that (B), (C), (D₁), and (D₂) imply (D₃). This follows from (5.1) and (D₂), since \( ab \cdot bc \cdot ca \) implies \( abc, bca, \) or \( cab \).

8. **Medians.** Some of the work in this section is similar to previous work (see, for example, [3, p. 137]). It has not, however, been previously shown that the results can be made independent of Postulate O₂ below.

Consider a set \( S \) closed under a ternary operation \( (a, b, c) \), called the median of \( a \), \( b \), and \( c \), satisfying the following postulates:

(M) \( (a, a, b) = a \).

(N) \(( (a, b, c), (a, b, d), e ) = ( (c, d, e), a, b ) \).

The following theorem is proved by setting \( c = d \) in (N) and applying (M). We have (8.2) as a corollary.
(8.1) \((a, b, c) = (c, a, b)\).

(8.2) \((b, a, a) = (a, b, a) = (a, a, b) = a\).

The next theorem is used freely in the work to follow.

(8.3) The median \((a, b, c)\) is invariant under permutations of \(a, b,\) and \(c\).

**Proof.** In view of (8.1), it is sufficient to show \((a, b, c) = (b, a, c)\). This follows from the equalities

\[
(a, b, c) = ((b, a, a), (b, a, b), c) = ((a, b, c), b, a)
= ((a, b, c), (a, b, b), a) = ((c, b, a), a, b)
= ((b, a, c), a, b) = ((a, b, b), (a, b, a), c) = (b, a, c).
\]

The next theorem is a direct consequence of (N) and (8.2).

(8.4) \(((a, b, c), a, b) = (a, b, c)\).

(8.5) \(((a, x, b), (b, x, c), (c, x, a)) = ((x, a, c), b, x)\).

**Proof.** By (N) and (8.4) both expressions equal \(((a, c, (c, x, a)), b, x)\).

As a corollary we have

(8.6) \(((x, a, c), b, x)\) is invariant under permutations of \(a, b,\) and \(c\).

(8.7) \(((a, b, c), (a, b, d), (a, b, e)) = ((c, d, e), a, b)\).

**Proof.** Using (N) and (8.6), we have

\[
((a, b, c), (a, b, d), (a, b, e))
= (((a, b, d), (a, b, e), a), ((a, b, d), (a, b, e), b), c)
= (((d, e, a), a, b), ((d, e, b), a, b), c)
= (((d, b, a), a, e), ((d, a, b), e, b), c)
= ((a, b, c), (d, b, a), e) = ((c, d, e), a, b).
\]

**Definition.** We say \(x\) is between \(a\) and \(b\) and write \(axb\) if and only if \(x = (a, x, b)\). Extensions of this notation such as \(abcd\) are made as in §4.

The next two theorems are immediate.

(8.8) \(abc \rightarrow cba\)

(8.9) \(abc \cdot acb \leftrightarrow b = c\).

Hence \(aab\) holds and \(aba \leftrightarrow a=b\).
\[(8.10)\] \[abc \cdot bcd \cdot ade \rightarrow bce.\]

**Proof.** We have \[(b, c, e) = ((a, b, c), (b, c, d), e) = ((a, d, e), b, c) = (d, b, c) = c.\]

\[(8.11)\] \[abc \cdot abd \cdot ced \rightarrow abe.\]

**Proof.** We have \[(a, b, e) = (a, b, (c, d, e)) = ((a, b, c), (a, b, d), e) = (b, b, e) = b.\]

The implications just established are found in \([4]\) under the labels \(\alpha, \beta, T_4,\) and \(T_7.\) We have from §6 of that paper the following as consequence.

\[(8.12)\] \[abc \cdot acd \rightarrow abed\]

\[(8.13)\] Given \(a, b,\) and \(c,\) there is a unique element \(x\) such that \(axb \cdot bxc \cdot cxa.\)

**Proof.** From (8.4), \((a, b, c)\) satisfies these betweenness relations.

If the relations hold for both \(x\) and \(y\) then \(x = (x, x, y) = (a, x, b),\)

\((a, x, c), y) = ((b, c, y), a, x) = (y, a, x)\) and, similarly, \(y = (y, a, x).\)

If we define \((a, b)\) as the set of all \(x\) such that \(x = (a, x, b)\) it is not difficult to prove the following.

\[(8.14)\] Postulates M and N imply Postulates S and T.

9. **Tree medians.** Consider Postulates M, N, and:

\((O_1)\) \([(x, a, b), c, x) = (x, a, c)\) or \((x, b, c).\)

It is clear from (8.6) and (8.7) that this postulate can be restated as follows. The elements \((a, x, b), (b, x, c),\) and \((c, x, a)\) are not distinct.

We wish to show these postulates characterize trees. By (5.10), (5.11), and (5.12), Postulates B, C, and \(D_1\) imply Postulates M, N, and \(O_1.\) Conversely, by (8.9) and (8.13), (M), (N), and \((O_1)\) imply (B) and (\(D_1)\). To show (C) also holds let \(b = (a, b, c), d = (b, d, e),\) and \(b \neq (c, b, d).\) By (8.2), \(b \neq d.\) By \((O_1), b = (b, d, b) = ((b, c, a), d, b) = (b, a, d).\) Hence \(b = (b, a, d), e, b) = (b, a, e)\) or \((b, d, e).\) To assume the latter gives the contradiction \(b = d.\) Hence \(b = (b, a, e)\) and \(eba\) holds. By (5.8), the following holds.

\[(9.1)\] Postulates M, N, and \(O_1\) characterize trees.

10. **Lattice medians.** Consider Postulates M, N, and:

\((O_2)\) There are elements 0 and \(I\) in \(S\) such that for all \(a\) in \(S,\)

\(a = (0, a, I).\)

In view of (8.3), the following is a restatement of a known theorem \([3, p. 137].\)
(10.1) Distributive lattices with bounds 0 and I are characterized by Postulates M, N, and O. This is achieved by use of the relations \( ab = (a, 0, b) \), \( a + b = (a, I, b) \), and \( (a, b, c) = ab + bc + ca \).

It is possible to characterize these lattices by means of only two postulates based on medians (see [6]).

11. **Chain medians.** Results in this section are similar to those in §7. We consider the postulate

\[(O_5) \text{ If } (a, b, c) \neq a \text{ or } c, \text{ then for all } x \text{ either } (a, x, b) \text{ or } (b, x, c) = b.\]

A simple four element example shows that the following cannot replace \((O_5)\) in (11.2). Postulates M, N, and \(O_5\) imply

\[(11.1) \quad (a, b, c) = a, b, \text{ or } c.\]

**Proof.** Let \( x \) in \((O_5)\) be \((a, b, c)\) and apply (8.4).

\[(11.2) \text{ Postulates M, N, and } O_5 \text{ characterize chains.}\]

**Proof.** Assume (M), (N), and (O_5). Postulates B and D_3 follow by (8.9) and (11.1). By (7.1), we have a chain if Postulate C holds. Assume \( abc \cdot bde \). If \( eba \), (C) holds. If \( bea \), we have in turn \( bdea, cbde, cbd, \) and (C) holds. But if neither \( eba \) or \( bea \) holds, we have

\[(11.3) \text{ Postulates M, N, O}_1, \text{ and } O_2 \text{ imply Postulate } O_2.\]

**Proof.** Since \((a, 0, b) = ((0, a, b), I, 0) = (0, a, I) \text{ or } (0, b, I)\), we have, say, \((a, 0, b) = a\). Then \((a, b, c) = ((0, b, a), (0, b, b), c) = ((a, b, c), 0, b) = (b, a, 0) \text{ or } (b, c, 0)\). Hence \((a, b, c) = a, b, \text{ or } c.\)

If \((a, b, c) = b, b = (b, x, b) = ((b, a, c), x, b) = (b, a, x) \text{ or } (b, c, x) \text{ and (O}_2\) is established.

12. **Median latticoids.** It is interesting that trees and bounded distributive lattices have an algebra with many properties in common. Assume Postulates M and N hold for a set of elements \( S. \) Let 0 and I be elements arbitrarily chosen from \( S. \) Define \( a + b \) as \((a, I, b)\) and \( ab \) as \((a, 0, b)\). \( S \) is closed under these operations and
properties V and W are immediate. Properties X and Y follow respectively from (8.6) and (N).

(V) The idempotent laws hold.
(W) The commutative laws hold.
(X) The associative laws hold.
(Y) The distributive laws hold and hence \( a + ab = a(a + b) \).
(Z) Elements 0 and I have the properties that, for all \( a \) and \( b \) in \( S \), \( a0 = 0 \), \( a + I = I \), and \( a + 0 = Ia = I(a + ab) \).

Proof. All properties except the last are immediate. The last follows from \((I, 0, (a, I, (a, 0, b))) = (I, 0, (a, b, (a, 0, I))) = (I, 0, a), b, (I, 0, (a, 0, I))) = ((I, 0, a), b, (a, 0, I)) = (I, 0, a)\).

Bibliography


Washington University