RIEMANN'S METHOD AND THE PROBLEM OF CAUCHY.
II. THE WAVE EQUATION IN n DIMENSIONS

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1. Introduction. In a recent paper Riemann’s method for the solution of the problem of Cauchy for a linear hyperbolic partial differential equation \( L(u) = 0 \) of second order for one unknown function \( u \) of two independent variables \( x, y \) was modified by the introduction of a line integral \( I_1 = \int \{ Bdx - A dy \} \) vanishing on closed paths. Here \( A \) and \( B \) are bilinear forms in the partial derivatives \( u_x, u_y, v_x, v_y \); and \( v \), the resolvent, is a properly chosen solution (analogous to Riemann’s function) of an associate equation \( M(v) = 0 \), the counterpart to the adjoint equation.

This modification opened the way to an extension of Riemann’s method to the wave equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = 0,
\]

in two dimensions. The line integral \( I_1 \) was replaced by an integral \( I_2 \) vanishing on closed surfaces and the associate equation \( M(v) = 0 \) turned out to be the Euler-Poisson equation

\[
\frac{1}{2} M(v) = v_{\alpha \beta} + \frac{1}{\alpha - \beta} \left( v_\alpha - v_\beta \right) = 0,
\]

with the resolvent

\[
v = \alpha + \beta + 2\left( (i - \alpha)(i - \beta) \right)^{1/2}
\]

taking over the role of Riemann’s function.

In the present paper the authors extend this method to the wave equation

\[
\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} - \frac{\partial^2 u}{\partial t^2} = 0,
\]

in \( n \) dimensions, \( n \geq 2 \), with, as might be expected, an \( n \)-dimensional integral \( I_n \), which vanishes over closed \( n \)-dimensional surfaces bounding \( (n+1) \)-dimensional volumes, replacing \( I_1 \) and \( I_2 \). The associate...
equation is now
\[ M(v) = v_{n\beta} + \frac{(n-1)/2}{\alpha - \beta} (v_{\alpha} - v_{\beta}) = 0, \]
and the resolvent is
\[ v = (i - \alpha)^{(n-1)/2} (i - \beta)^{(n-1)/2}. \]

2. The Laplacian \( \Delta_{n} \) in polar coordinates. Consider the generalization to \( n \) dimensions of the well known space polar coordinate system \( \phi, \theta, r \), in three dimensions, where
\[
\begin{align*}
  x &= r \cos \phi \sin \theta, \\
  y &= r \sin \phi \sin \theta, \\
  z &= r \cos \theta,
\end{align*}
\]
and if we write
\[
\begin{align*}
  y_1 &= \phi, \\
  y_2 &= \theta_1, \\
  &\vdots \\
  y_{n-1} &= \theta_{n-2}, \\
  y_n &= r,
\end{align*}
\]
and if we write
\[
\begin{align*}
  g_{11} &= r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}, \\
  g_{22} &= r^2 \sin^2 \theta_2 \cdots \sin^2 \theta_{n-2}, \\
  &\vdots \\
  g_{n,n-1} &= r^2, \\
  g_{nn} &= 1.
\end{align*}
\]
we shall have
\[
\Delta_{n} u = \frac{1}{g^{1/2}} \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left( \frac{g^{1/2}}{g_{ii}} u_{yi} \right),
\]
and if we set
\[
 s_{i-1} = g^{1/2} / r^{n-2} g_{ii}, \quad i = 1, \ldots, n-1,
\]
so that
\[ f_0 = \csc \theta_1 \sin \theta_2 \sin^2 \theta_4 \cdots \sin^{n-4} \theta_{n-2}; \]
\[ f_1 = \sin \theta_1 \sin \theta_2 \sin^2 \theta_4 \cdots \sin^{n-4} \theta_{n-2}; \]
\[ f_2 = \sin \theta_1 \sin^2 \theta_2 \sin \theta_3 \sin^2 \theta_4 \cdots \sin^{n-4} \theta_{n-2}; \]
\[ f_3 = \sin \theta_1 \sin^2 \theta_2 \sin^3 \theta_3 \sin \theta_4 \sin^2 \theta_6 \cdots \sin^{n-4} \theta_{n-2}; \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ f_{n-3} = \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-3} \theta_{n-3} \sin^{n-4} \theta_{n-2}; \]
\[ f_{n-2} = \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2}; \]

we find

\[ \Delta_3 u = f^{-1} \left[ \frac{\partial}{\partial \phi} \left( \frac{f_0}{r^2} u_{\phi} \right) \right] + \sum_{i=1}^{n-2} \frac{\partial}{\partial \theta_i} \left( \frac{f_i}{r^2} u_{\theta_i} \right) ] + u_{rr} + \frac{n-1}{r} u_r, \]

provided we write

\[ f = \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2} = f_{n-2}. \]

We note in passing that the element of \((n-1)\)-dimensional area on the unit sphere \(r = 1\) is

\[ d\omega_n = f d\phi d\theta_1 \cdots d\theta_{n-2}, \]

and the \((n-1)\)-dimensional area of the unit sphere is

\[ \omega_n = \int_{0}^{\pi} \cdots \int_{0}^{\pi} \int_{0}^{2\pi} f d\phi d\theta_1 \cdots d\theta_{n-2} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \]

3. A fundamental identity. Starting with the polar coordinates \(\phi, \theta_1, \cdots, \theta_{n-2}, r\) in \(n\)-dimensional space we introduce coordinates \(\alpha, \beta, \phi, \theta_1, \cdots, \theta_{n-2}\) in \((n+1)\)-dimensional space-time by setting

\[ \alpha = t + r, \quad \beta = t - r, \]

and term \(\alpha, \beta\) characteristic coordinates, inasmuch as \(\alpha=\text{const.}, \beta=\text{const.}\) are characteristic half-cones for the wave equation

\[ L(u) = \frac{1}{4} (u_{tt} - \Delta_2 u) = 0. \]

In these coordinates the operator \(L(u)\) takes the form

\[ L(u) = \left[ u_{\alpha \beta} - \frac{(n-1)/2}{\alpha - \beta} (u_\alpha - u_\beta) \right] f \]

\[ - (\alpha - \beta)^{-2} \left[ \frac{\partial}{\partial \phi} (f_{0} u_{\phi}) + \sum_{i=1}^{n-2} \frac{\partial}{\partial \theta_i} (f_i u_{\theta_i}) \right], \]
with which we associate the operator

\[ M(v) = v_{\alpha \beta} + \frac{(n-1)/2}{\alpha - \beta} (v_{\alpha} - v_{\beta}). \]

If we write

\[ A = \int u_{\alpha} v_{\beta}, \quad B = -\int u_{\alpha} v_{\alpha}, \]

\[ \Phi = \int_0^1 \frac{v_{\alpha} - v_{\beta}}{(\alpha - \beta)^2} \dot{u}_{\Phi}, \quad \Theta_j = \int_0^1 \frac{v_{\alpha} - v_{\beta}}{(\alpha - \beta)^2} \dot{u}_{\Theta_j}, \]

\( j=1, \ldots, n-2, \) a simple calculation shows that (note that \( v = v(\alpha, \beta) \))

\[ (5) \quad A_{\alpha} + B_{\beta} + \Phi_{\alpha} + \sum_{j=1}^{n-2} \frac{\partial \Theta_j}{\partial \theta_j} = (v_{\beta} - v_{\alpha})L(u) + (u_{\beta} - u_{\alpha})fM(v). \]

This identity plays the role of a Green's identity⁴ in our investigation, the part of the adjoint equation being taken over by the associate equation \( M(v) = 0. \)

According to the generalized Green's theorem, the surface integral

\[ I_n = \int_{S_n} \{ A d\beta d\dot{\phi} d\theta_1 \cdots d\theta_{n-2} + B d\alpha d\dot{\phi} d\theta_1 \cdots d\theta_{n-2} \]

\[ + \Phi d\alpha d\beta d\theta_1 \cdots d\theta_{n-2} + \cdots + \Theta_{n-2} d\alpha d\beta d\phi \cdots d\theta_{n-2} \} \]

\[ = \int_{S_n} \{ A v_{\alpha} + B v_{\beta} + \Phi v_{\alpha} + \cdots + \Theta_{n-2} v_{\alpha} \} dS_n \]

(where \( v_{\alpha}, v_{\beta}, \cdots \) are the components of the unit outer normal to \( S_n \)) when extended around a closed \( n \)-dimensional surface \( S_n \) bounding an \( (n+1) \)-dimensional volume \( V_{n+1} \) can be expressed as a volume integral over \( V_{n+1} \), namely

\[ \int_{V_{n+1}} \left( A_{\alpha} + B_{\beta} + \Phi_{\alpha} + \sum_{j=1}^{n-2} \frac{\partial \Theta_j}{\partial \theta_j} \right) d\alpha d\beta d\phi d\theta_1 \cdots d\theta_{n-2}. \]

The following lemma is now obvious.

**Lemma.** The surface integral \( I_n \), taken around a closed \( n \)-dimensional surface \( S_n \), vanishes whenever \( u, v \) are regular solutions of \( L(u) = 0 \), and its associate equation \( M(v) = 0 \), respectively.

It is worth while to note that each of $A$, $B$, $\Phi$, $\Theta_1$, \ldots, $\Theta_{n-2}$ is a bilinear form in the partial derivatives of first order of $u$ and $v$ with respect to $\alpha$, $\beta$, $\phi$, $\theta_1$, \ldots, $\theta_{n-2}$.

4. The problem of Cauchy. As Cauchy data on the hyperplane $t=0$ in $(n+1)$-dimensional space-time we take

\begin{align*}
u(x_1, \ldots, x_n, 0) &= u^0(x_1, \ldots, x_n), \\
\nu_t(x_1, \ldots, x_n, 0) &= u^1(x_1, \ldots, x_n),
\end{align*}

the functions $u^0$, $u^1$ being given in advance. Let $P_i$ denote the point with coordinates $(x_1, \ldots, x_n, t)$ in space-time. The solution of the problem of Cauchy requires the value $u(P_i)$ of the solution $u$ of $L(u) = 0$ to be expressed in terms of the initial data $u^0$, $u^1$ carried by the part of the initial hyperplane $t=0$ contained within the ("retrograde") characteristic half-cone with vertex at $P_i$, i.e., in terms of the initial data assigned to the points

\[(x_1 - \bar{x}_1)^2 + \cdots + (x_n - \bar{x}_n)^2 \leq \bar{t}, \quad t = 0.\]

We assume $t > 0$ and consider the $(n+1)$-dimensional conical volume $C$ bounded in space-time by the characteristic hypercone with vertex at $P_i$ and the initial hyperplane $t=0$. The axis of $C$ is the straight line $P_0P_i$ in space-time traced out by $P_i$ as $t$ ranges from 0 to $\bar{t}$. If at each point $P_i$ we introduce polar coordinates $\phi$, $\theta_1$, \ldots, $\theta_{n-2}$, $r$ with pole at $P_i$, the conical volume $C$ is described by the inequalities

\[C: \quad 0 \leq \phi < 2\pi, \quad 0 \leq \theta_j \leq \pi, \quad 0 \leq r \leq \bar{t} - t, \quad 0 \leq t \leq \bar{t} \quad (j = 1, \ldots, n - 2).\]

When we take $\alpha$, $\beta$, $\phi$, $\theta_1$, \ldots, $\theta_{n-2}$ as rectangular coordinates in a second $(n+1)$-dimensional space, $C$ appears as a "wedge"

\[W: \quad 0 \leq \alpha \leq \bar{t}, \quad -\alpha \leq \beta \leq + \alpha, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \theta_j \leq \pi \quad (j = 1, \ldots, n - 2).\]

That part of the boundary of $C$ formed by the mantle of the characteristic hypercone becomes the face $\alpha = \bar{t}$ of $W$; the base $t=0$ of $C$ is represented by the face $\beta = -\alpha$ of $W$; and the axis $P_0P_i$ of $C$ by the face $\beta = \alpha$ of $W$. The vertex $P_i$ of $C$ appears as the edge $\alpha = \beta = \bar{t}$ of $W$; the periphery of the base of $C$ (the intersection of the initial plane with the characteristic hypercone) is replaced by the edge $\alpha = -\beta = t$ of $W$; and center $P_0$ of the base of $C$ by the edge $\alpha = \beta = 0$ of $W$.

To reformulate the problem of Cauchy in $(\alpha, \beta, \phi, \theta_1$, \ldots, $\theta_{n-2})$-coordinates, one may use the transformation of variables $\alpha = \alpha_1 + \beta_1$, $\beta = \alpha_1 - \beta_1$, $\phi = \phi_1$, $\theta_j = \theta_j_1$ in the conical volume $C$.
space we observe that the carrier \( t=0 \) becomes the hyperplane \( \beta = -\alpha \) upon which, from (3), we assign

\[ u_\phi = u_\phi, \quad u_\phi = u_\phi, \quad u_\alpha = (u_\tau + u_\alpha)/2, \quad u_\beta = -\frac{u_\tau - u_\alpha}{2} \]

as initial data. One would accordingly seek an expression for the value of the solution \( u \) of \( L(u) = 0 \), for \( L(u) \) as defined in (4), along the edge \( \alpha = \beta = \tau \) of \( W \) in terms of the above initial data carried by the face \( \beta = -\alpha \) of \( W \).

To solve the problem of Cauchy as originally formulated we apply the lemma of the preceding section to the closed surface \( S_n \) which is the boundary of the wedge \( W \) and obtain

\[
\int_{\beta = \alpha}^{\beta = -\alpha} I_n + I_n + I_n + \left( I_n + I_n \right) + \sum_{\phi = 0}^{2\pi} \left( I_n + I_n \right) = 0.
\]

For single-valued solutions, \( u \) must be periodic of period \( 2\pi \) in \( \phi \) and it follows from the definition of \( \Theta \) that

\[
I_n + I_n = 0,
\]

since the external normals to \( S_n \) have opposite directions on the faces \( \phi = 0, \phi = 2\pi \). Since \( \Theta_j \) involves \( f_j \), and \( f_j \) contains \( \sin \theta_j \) as a factor for \( j = 1, \cdots, n-2 \), it is clear that

\[
I_n = I_n = 0,
\]

and the above result simplifies to

\[
I_n + I_n + I_n = 0.
\]

The integration of \( I_n \) in (6) over \( S_n \) yields

\[
\int_0^I \int_{\omega_n} \left[ -A + B \right]_{\beta = \alpha} f^{-1} d\omega_n d\alpha - \int_0^I \int_{\omega_n} \left[ A + B \right]_{\beta = -\alpha} f^{-1} d\omega_n d\alpha
\]

\[
+ \int_0^I \int_{\omega_n} A_{\alpha = -1} f^{-1} d\omega_n d\beta = 0,
\]

and when we employ the definitions of \( A \) and \( B \), we find

\[
- \int_0^I \int_{\omega_n} \left[ u_\alpha v_\alpha + u_\beta v_\beta \right]_{\beta = \alpha} d\omega_n d\alpha
\]

\[
+ \int_0^I \int_{\omega_n} \left[ u_\alpha v_\alpha - u_\beta v_\beta \right]_{\beta = -\alpha} d\omega_n d\alpha + \int_0^I \int_{\omega_n} u_\beta v_\beta \int_{\alpha = -1} d\omega_n d\beta = 0.
\]
Up to this point $v$ has been any solution of the associate equation $M(v) = 0$. For $v$ we now take the special solution
\[ v = (\tilde{t} - \alpha)^{(n-1)/2}(\tilde{t} - \beta)^{(n-1)/2}, \quad n \geq 2. \]
This solution, termed the resolvent, is obtained by applying the ordinary method of separation of variables to $M(v) = 0$ and plays the role of "Riemann's function." It is convenient to observe that
\[ \beta = \alpha \text{ implies } r = 0, \quad \alpha = t, \]
\[ \beta = - \alpha \text{ implies } t = 0, \quad \alpha = r, \]
\[ \alpha = \tilde{t} \text{ implies } v_\beta = 0. \]
More precisely, the last relations hold for $n \geq 3$, and (8) holds as a result of integrating the fundamental identity (5) over the "wedge" $W$, all integrals involved being proper integrals. However, if $n = 2$ then $v_\beta$ is infinite on $\alpha = \tilde{t}$ and in order to obtain (8)—where improper integrals now appear—it is necessary to integrate first the identity (5) in $(\alpha, \beta, \phi)$-space over the smaller "wedge" $W_\varepsilon$, whose cross section in the $\alpha\beta$-plane is bounded by the four straight lines
\[ \alpha = \beta, \quad \alpha = - \beta, \quad \beta = t - \varepsilon, \quad \beta = - t + \varepsilon, \]
where $0 < \varepsilon < \varepsilon < \tilde{t}$. Passing to the limit, letting $\varepsilon \to 0$ first, and afterwards letting $\varepsilon \to 0$, yields (8).

Thus the last term in (8) drops out altogether, eliminating the need for prescribed data on the characteristic half-cone, and the result is
\[ \int_0^t \int_{\omega_n} (\tilde{t} - t)^{n-2} \mathcal{U}_t \left| \begin{array}{c} \omega_n \alpha = 0 \\ \mathcal{U}_n \end{array} ight. d\omega_n dt \\
= \int_0^t \int_{\omega_n} \left[ (P^2 - r^2)^{(n-3)/2} \mathcal{U}_n + (P^2 - r^2)^{(n-3)/2} \mathcal{U}_n \right] d\omega_n dr, \]
where the integration on the left is performed on the axis of the cone.

* Compare G. Darboux, loc. cit., p. 70, for $n = 2$. 
C. Since

\[
\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} dt_{m-1} \left[ \int_0^{t_m} f(t_m) \, dt_m \right] = \int_0^t \frac{(t - \iota)^{m-1}}{(m - 1)!} f(t) \, dt,
\]

it follows that the preceding relation may be differentiated at least \( n - 1 \) times with respect to \( \iota \). Differentiating \( n - 2 \) times with respect to \( \iota \) yields the final formula:

\[
u(\Pi) = \nu(\Pi_0) + \frac{1}{(n - 2) \omega_n} \frac{\partial^{n-2}}{\partial \iota^{n-2}} \int_0^t \int_{\omega_n} \left[ \left( \frac{n-3}{2} \right) \nu^2 + \frac{(n-2)}{2} \nu \right] \, d\omega_n \, dr.
\]

In the present notation, the usual formula\(^7\) for the solution of the Cauchy problem considered above may be written

\[
u(\Pi) = \frac{1}{(n - 2) \omega_n} \frac{\partial^{n-1}}{\partial \iota^{n-1}} \int_0^t \int_{\omega_n} \left( \frac{n-3}{2} \right) \nu^2 + \left( \frac{n-2}{2} \right) \nu \right] \, d\omega_n \, dr.
\]

The two formulas for \( \nu(\Pi) \) are easily seen to coincide,\(^8\) upon differentiating once with respect to \( \iota \) the first integral on the right-hand side of (10). This differentiation may be carried out directly under the integral sign if one first sets \( r = \iota p \). A subsequent integration by parts then yields the result.

In conclusion, the above argument shows the uniqueness of the solution of Cauchy's problem. More precisely, if the Cauchy problem considered has a solution \( \nu \) which possesses continuous second derivatives on \( t > 0 \) and continuous first derivatives on \( t \geq 0 \), then \( \nu \) is given by formula (9).

\(^8\) See M. H. Martin, loc. cit., page 244, for the case \( n = 2 \).