RIEMANN'S METHOD AND THE PROBLEM OF CAUCHY.
II. THE WAVE EQUATION IN \( n \) DIMENSIONS\(^1\)

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1. Introduction. In a recent paper\(^2\) Riemann's method for the solution of the problem of Cauchy for a linear hyperbolic partial differential equation \( L(u) = 0 \) of second order for one unknown function \( u \) of two independent variables \( x, y \) was modified by the introduction of a line integral \( I_1 = \int \{ Bdx - Ady \} \) vanishing on closed paths. Here \( A \) and \( B \) are bilinear forms in the partial derivatives \( u_x, u_y, v_x, v_y \); and \( v \), the resolvent, is a properly chosen solution (analogous to Riemann's function) of an associate equation \( M(v) = 0 \), the counterpart to the adjoint equation.

This modification opened the way to an extension of Riemann's method to the wave equation

\[
\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u - \frac{\partial^2}{\partial t^2} u = 0,
\]

in two dimensions. The line integral \( I_1 \) was replaced by an integral \( I_2 \) vanishing on closed surfaces and the associate equation \( M(v) = 0 \) turned out to be the Euler-Poisson equation\(^3\)

\[
M(v) = v_{\alpha \beta} + \frac{1}{\alpha - \beta} (v_{\alpha} - v_{\beta}) = 0,
\]

with the resolvent

\[
v = \alpha + \beta + 2[(\tilde{t} - \alpha)(\tilde{t} - \beta)]^{1/2}
\]

taking over the role of Riemann's function.

In the present paper the authors extend this method to the wave equation

\[
\frac{\partial^2}{\partial x_1^2} u + \cdots + \frac{\partial^2}{\partial x_n^2} u - \frac{\partial^2}{\partial t^2} u = 0,
\]

in \( n \) dimensions, \( n \geq 2 \), with, as might be expected, an \( n \)-dimensional integral \( I_n \), which vanishes over closed \( n \)-dimensional surfaces bounding \((n+1)\)-dimensional volumes, replacing \( I_1 \) and \( I_2 \). The associate

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equation is now
\[ M(v) = v_{a\beta} + \frac{(n-1)/2}{\alpha - \beta} (v_{a\alpha} - v_{a\beta}) = 0, \]
and the resolvent is
\[ v = (\mathbb{I} - \lambda)^{(n-1)/2}(\mathbb{I} - \mathbb{B})^{(n-1)/2}. \]

2. The Laplacian $\Delta_2 u = u_{x_1x_1} + \cdots + u_{x_nx_n}$ in polar coordinates. Consider the generalization to $n$ dimensions of the well known space polar coordinate system $\phi, \theta, r$, in three dimensions, where
\[
\begin{align*}
  x &= r \cos \phi \sin \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \theta, \\
  0 \leq \phi < 2\pi, \quad 0 \leq \theta \leq \pi, \quad r \geq 0,
\end{align*}
\]
that is, coordinates $\phi, \theta_1, \cdots, \theta_{n-2}, r$ with
\[
\begin{align*}
  x_1 &= r \cos \phi \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2}, \quad 0 \leq \phi < 2\pi, \\
  x_2 &= r \sin \phi \sin \theta_1 \sin \theta_2 \cdots \sin \theta_{n-2}, \quad 0 \leq \theta_1 \leq \pi, \\
  x_3 &= r \cos \phi \sin \theta_2 \cdots \sin \theta_{n-2}, \quad 0 \leq \theta_2 \leq \pi, \\
  &\quad \vdots \quad \vdots \quad \vdots
\end{align*}
\]
\[ x_n = r \cos \theta_{n-2}, \quad r \geq 0. \]
The element of arc is given by
\[
\begin{align*}
  ds^2 &= r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2} d\phi^2 + r^2 \sin^2 \theta_2 \cdots \sin^2 \theta_{n-2} d\theta_1^2 + \cdots \nonumber \\
  &\quad + r^2 d\theta_{n-2}^2 + dr^2,
\end{align*}
\]
and if we write
\[
\begin{align*}
  y_1 &= \phi, \quad y_2 = \theta_1, \cdots, \quad y_{n-1} = \theta_{n-2}, \quad y_n = r, \\
  g_{11} &= r^2 \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2}, \\
  g_{22} &= r^2 \sin^2 \theta_2 \cdots \sin^2 \theta_{n-2}, \cdots, \quad g_{n-1,n-1} = r^2, \quad g_{nn} = 1. 
\end{align*}
\]
we shall have
\[
\Delta_2 u = \frac{1}{g^{1/2}} \sum_{i=1}^{n} \frac{\partial}{\partial y_i} \left( \frac{g^{1/2}}{g_{ii}} u_{y_i} \right),
\]
\[ g^{1/2} = r^{n-1} \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-1}. \]
If we set $f_{i-1} = g^{1/2}/r^{n-2}g_{ii}$, $i = 1, \cdots, n-1$, so that
\[ f_0 = \csc \theta_1 \sin \theta_2 \sin^2 \theta_4 \cdots \sin^{n-4} \theta_{n-2}; \]

\[ f_1 = \sin \theta_1 \sin \theta_2 \sin^2 \theta_4 \cdots \sin^{n-4} \theta_{n-2}; \]

\[ f_2 = \sin \theta_1 \sin^2 \theta_2 \sin \theta_3 \sin^2 \theta_4 \cdots \sin^{n-4} \theta_{n-2}; \]

\[ f_n = \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-3} \sin^{n-4} \theta_{n-2}; \]

\[ f_{n-1} = \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2}; \]

we find

\[ \Delta_z u = f^{-1} \left[ \frac{\partial}{\partial \phi} \left( \frac{f_0}{r^2} u_{\phi} \right) + \sum_{i=1}^{n-2} \frac{\partial}{\partial \theta_i} \left( \frac{f_i}{r^2} u_{\theta_i} \right) \right] + u_{rr} + \frac{n - 1}{r} u_r, \]

provided we write

\[ f = \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2} = f_{n-2}. \]

We note in passing that the element of \((n - 1)\)-dimensional area on the unit sphere \(r = 1\) is

\[ d\omega_n = f d\phi d\theta_1 \cdots d\theta_{n-2}, \]

and the \((n - 1)\)-dimensional area of the unit sphere is

\[ \omega_n = \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} f d\phi d\theta_1 \cdots d\theta_{n-2} = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \]

3. A fundamental identity. Starting with the polar coordinates \(\phi, \theta_1, \cdots, \theta_{n-2}, r\) in \(n\)-dimensional space we introduce coordinates \(\alpha, \beta, \phi, \theta_1, \cdots, \theta_{n-2}\) in \((n + 1)\)-dimensional space-time by setting

\[ \alpha = t + r, \quad \beta = t - r, \]

and term \(\alpha, \beta\) characteristic coordinates, inasmuch as \(\alpha = \text{const.}, \beta = \text{const}\). are characteristic half-cones for the wave equation

\[ L(u) = \frac{1}{4} (u_{tt} - \Delta_z u) = 0. \]

In these coordinates the operator \(L(u)\) takes the form

\[ L(u) = \left[ u_{\alpha \beta} - \frac{(n - 1)/2}{\alpha - \beta} (u_\alpha - u_\beta) \right] f \]

\[ - (\alpha - \beta)^{-2} \left[ \frac{\partial}{\partial \phi} (f_0 u_\phi) + \sum_{i=1}^{n-2} \frac{\partial}{\partial \theta_i} (f_i u_{\theta_i}) \right], \]

\[ f = \sin \theta_1 \sin^2 \theta_2 \cdots \sin^{n-2} \theta_{n-2}. \]
with which we associate the operator

\[ M(v) = \nu_{\alpha\beta} + \frac{(n-1)/2}{\alpha - \beta} (\nu_{\alpha} - \nu_{\beta}). \]

If we write

\[ A = \int \nu_{\alpha\beta} \nu_{\beta}, \quad B = -\int \nu_{\alpha} \nu_{\alpha}, \]

\[ \Phi = \int_0^{\alpha - \beta} \nu_{\alpha} - \nu_{\beta}, \quad \Theta_i = \int_0^{\alpha - \beta} \nu_{\alpha} - \nu_{\beta}, \]

\[ j = 1, \ldots, n-2, \] a simple calculation shows that (note that \( \nu = \nu(\alpha, \beta) \))

\[ (5) \quad A_{\alpha} + B_{\beta} + \Phi_{\phi} + \sum_{j=1}^{n-2} \frac{\partial \Theta_j}{\partial \theta_j} = (\nu_{\beta} - \nu_{\alpha})L(u) + (\nu_{\alpha} - \nu_{\alpha})fM(v). \]

This identity plays the role of a Green's identity in our investigation, the part of the adjoint equation being taken over by the associate equation \( M(v) = 0 \).

According to the generalized Green's theorem, the surface integral

\[ I_n = \int_{S_n} \{ A d\beta d\phi d\theta_1 + \cdots + B d\alpha d\phi d\theta_1 + \cdots + \Phi d\alpha d\beta d\phi \cdots \} \]

\[ = \int_{S_n} \{ A\nu_{\alpha} + B\nu_{\beta} + \Phi \nu_{\phi} + \cdots + \Theta_{n-2} \nu_{n-2} \} dS_n \]

(6) \hspace{1cm} (\text{where} \, \nu_{\alpha}, \nu_{\beta}, \cdots \, \text{are the components of the unit outer normal to} \, S_n) \]

when extended around a closed \( n \)-dimensional surface \( S_n \) bounding an \((n+1)\)-dimensional volume \( V_{n+1} \) can be expressed as a volume integral over \( V_{n+1} \), namely

\[ \int_{V_{n+1}} \left( A_{\alpha} + B_{\beta} + \Phi_{\phi} + \sum_{j=1}^{n-2} \frac{\partial \Theta_j}{\partial \theta_j} \right) d\alpha d\beta d\phi d\theta_1 + \cdots + d\theta_{n-2}. \]

The following lemma is now obvious.

**Lemma.** The surface integral \( I_n \), taken around a closed \( n \)-dimensional surface \( S_n \), vanishes whenever \( u, v \) are regular solutions of \( L(u) = 0 \), and its associate equation \( M(v) = 0 \), respectively.

\[^4\text{Compare the "formule fondamentale" in the terminology of J. Hadamard, Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques, Paris, 1932, chapter II, esp. p. 83.}\]
It is worth while to note that each of $A$, $B$, $\Phi$, $\Theta_1$, \ldots, $\Theta_{n-2}$ is a bilinear form in the partial derivatives of first order of $u$ and $v$ with respect to $\alpha$, $\beta$, $\phi$, $\theta_1$, \ldots, $\theta_{n-2}$.

4. The problem of Cauchy. As Cauchy data on the hyperplane $t=0$ in $(n+1)$-dimensional space-time we take

$$u(x_1, \ldots, x_n, 0) = u^0(x_1, \ldots, x_n),$$

$$u_t(x_1, \ldots, x_n, 0) = u^1(x_1, \ldots, x_n),$$

the functions $u^0$, $u^1$ being given in advance. Let $P_i$ denote the point with coordinates $(x_1, \ldots, x_n, t)$ in space-time. The solution of the problem of Cauchy requires the value $u(P_i)$ of the solution $u$ of $L(u) = 0$ to be expressed in terms of the initial data $u^0$, $u^1$ carried by the part of the initial hyperplane $t=0$ contained within the ("retrograde") characteristic half-cone with vertex at $P_i$, i.e., in terms of the initial data assigned to the points

$$(x_1 - x_1)^2 + \cdots + (x_n - x_n)^2 \leq \vec{P}, \quad t = 0.$$

We assume $t>0$ and consider the $(n+1)$-dimensional conical volume $C$ bounded in space-time by the characteristic hypercone with vertex at $P_i$ and the initial hyperplane $t=0$. The axis of $C$ is the straight line $P_0P_i$ in space-time traced out by $P_i$ as $t$ ranges from 0 to $\vec{t}$. If at each point $P_i$ we introduce polar coordinates $\phi, \theta_1, \ldots, \theta_{n-2}, r$ with pole at $P_i$, the conical volume $C$ is described by the inequalities

$$C: \quad 0 \leq \phi < 2\pi, \quad 0 \leq \theta_j \leq \pi, \quad 0 \leq r \leq \vec{t} - t, \quad 0 \leq t \leq \vec{t}$$

$$(j = 1, \ldots, n - 2).$$

When we take $\alpha, \beta, \phi, \theta_1, \ldots, \theta_{n-2}$ as rectangular coordinates in a second $(n+1)$-dimensional space, $C$ appears as a "wedge"

$$W: \quad 0 \leq \alpha \leq \vec{t}, \quad -\alpha \leq \beta \leq +\alpha, \quad 0 \leq \phi < 2\pi, \quad 0 \leq \theta_j \leq \pi$$

$$(j = 1, \ldots, n - 2).$$

That part of the boundary of $C$ formed by the mantle of the characteristic hypercone becomes the face $\alpha=\vec{t}$ of $W$; the base $t=0$ of $C$ is represented by the face $\beta=-\alpha$ of $W$; and the axis $P_0P_i$ of $C$ by the face $\beta=\alpha$ of $W$. The vertex $P_i$ of $C$ appears as the edge $\alpha=\beta=\vec{t}$ of $W$; the periphery of the base of $C$ (the intersection of the initial plane with the characteristic hypercone) is replaced by the edge $\alpha=-\beta=\vec{t}$ of $W$; and center $P_0$ of the base of $C$ by the edge $\alpha=\beta=0$ of $W$.

To reformulate the problem of Cauchy in $(\alpha, \beta, \phi, \theta_1, \ldots, \theta_{n-2})$-

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* Compare M. H. Martin, loc. cit., p. 245.
space we observe that the carrier $t=0$ becomes the hyperplane $\beta = -\alpha$ upon which, from (3), we assign

$$u_\phi = u_\phi, \quad u_{\theta j} = u_{\theta j}, \quad u_a = (u_\theta + u_\phi)/2, \quad u_\beta = -(u_\theta - u_\phi)/2$$

as initial data. One would accordingly seek an expression for the value of the solution $u$ of $L(u) = 0$, for $L(u)$ as defined in (4), along the edge $\alpha = \beta = t$ of $W$ in terms of the above initial data carried by the face $\beta = -\alpha$ of $W$.

To solve the problem of Cauchy as originally formulated we apply the lemma of the preceding section to the closed surface $S_n$ which is the boundary of the wedge $W$ and obtain

$$I_n + I_n + I_n + \sum_{j=1}^{n-2} \left( I_n + I_n \right) = 0.$$

For single-valued solutions, $u$ must be periodic of period $2\pi$ in $\phi$ and it follows from the definition of $\Phi$ that

$$I_n + I_n = 0,$$

since the external normals to $S_n$ have opposite directions on the faces $\phi = 0, \phi = 2\pi$. Since $\Theta_j$ involves $f_j$, and $f_j$ contains $\sin \theta_j$ as a factor for $j = 1, \cdots, n-2$, it is clear that

$$I_n = I_n = 0,$$

and the above result simplifies to

$$I_n + I_n + I_n = 0.$$

The integration of $I_n$ in (6) over $S_n$ yields

$$\int_0^1 \int_{u_n} \left[-A + B\right]_{\beta = \alpha} f^{-1} \omega \, d\alpha + \int_0^1 \int_{u_n} \left[A + B\right]_{\beta = \alpha} f^{-1} \omega \, d\alpha$$

and when we employ the definitions of $A$ and $B$, we find

$$\int_0^1 \int_{u_n} \left[u_\alpha v_\alpha + u_\beta v_\beta\right]_{\beta = \alpha} \omega \, d\alpha$$

and

$$\int_0^1 \int_{u_n} \left[u_\alpha v_\alpha - u_\beta v_\beta\right]_{\beta = \alpha} \omega \, d\alpha + \int_0^1 \int_{u_n} u_\beta v_\beta \omega \, d\beta = 0.$$
Up to this point \( v \) has been any solution of the associate equation \( M(v) = 0 \). For \( v \) we now take the special solution\(^6\)

\[
v = (t - \alpha)^{(n-1)/2}(i - \beta)^{(n-1)/2};
\]

This solution, termed the *resolvent*, is obtained by applying the ordinary method of separation of variables to \( M(v) = 0 \) and plays the role of "Riemann's function." It is convenient to observe that

- \( \beta = \alpha \) implies \( \tau = 0, \alpha = t \),

\[
\frac{v_\alpha - v_\beta}{2} = 0, \quad \frac{v_\alpha + v_\beta}{2} = -\frac{n - 1}{2} (i - t)^{n-2},
\]

- \( \beta = -\alpha \) implies \( \tau = 0, \alpha = r \),

\[
\frac{v_\alpha - v_\beta}{2} = -\frac{n - 1}{2} (\tau^2 - \tau^2)^{(n-3)/2} r, \\
\frac{v_\alpha + v_\beta}{2} = -\frac{n - 1}{2} (\tau^2 - \tau^2)^{(n-3)/2} i,
\]

- \( \alpha = \bar{i} \) implies \( v_\beta = 0 \).

More precisely, the last relations hold for \( n \geq 3 \), and (8) holds as a result of integrating the fundamental identity (5) over the "wedge" \( W \), all integrals involved being proper integrals. However, if \( n = 2 \) then \( v_\beta \) is infinite on \( \alpha = \bar{i} \) and in order to obtain (8)—where improper integrals now appear—it is necessary to integrate first the identity (5) in \((\alpha, \beta, \phi)\)-space over the smaller "wedge" \( W_{e, n} \) whose cross section in the \( \alpha \beta \)-plane is bounded by the four straight lines

\[
\alpha = \beta, \quad \alpha = -\beta, \quad \beta = \bar{i} - \eta, \quad \alpha = \bar{i} - \eta,
\]

where \( 0 < \eta < \epsilon < \bar{i} \). Passing to the limit, letting \( \eta \to 0 \) first, and afterwards letting \( \epsilon \to 0 \), yields (8).

Thus the last term in (8) drops out altogether, eliminating the need for prescribed data on the characteristic half-cone, and the result is

\[
\int_0^I \int_{\omega_n} (\bar{i} - t)^{n-2} \frac{d\omega dt}{r=0} \\
= \int_0^I \int_{\omega_n} [(\bar{i}^2 - \tau^2)^{(n-3)/2} \tau_0^0 + (\bar{i}^2 - \tau^2)^{(n-3)/2} \tau \cdot \eta^1] d\omega dr,
\]

where the integration on the left is performed on the axis of the cone.

\(^6\) Compare G. Darboux, loc. cit., p. 70, for \( n = 2 \).
it follows that the preceding relation may be differentiated at least \( n - 1 \) times with respect to \( t \). Differentiating \( n - 2 \) times with respect to \( t \) yields the final formula:

\[
\frac{\partial^{n-2}}{\partial t^{n-2}} \int_0^t \int_{\omega_n} \left[ (t^2 - r^2)^{(n-3)/2} t u_r^2 + (t^2 - r^2)^{(n-2)/2} r u^1 \right] d\omega_n dr.
\]

In the present notation, the usual formula\(^7\) for the solution of the Cauchy problem considered above may be written

\[
u(P_t) = \frac{1}{(n-2)\omega_n} \int_0^t \int_{\omega_n} (t^2 - r^2)^{(n-3)/2} r u^1 d\omega_n dr.
\]

The two formulas for \( \nu(P_t) \) are easily seen to coincide,\(^8\) upon differentiating once with respect to \( t \) the first integral on the right-hand side of (10). This differentiation may be carried out directly under the integral sign if one first sets \( r = \tilde{r}p \). A subsequent integration by parts then yields the result.

In conclusion, the above argument shows the uniqueness of the solution of Cauchy’s problem. More precisely, if the Cauchy problem considered has a solution \( u \) which possesses continuous second derivatives on \( t > 0 \) and continuous first derivatives on \( t \geq 0 \), then \( u \) is given by formula (9).

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\(^8\) See M. H. Martin, loc. cit., page 244, for the case \( n = 2 \).