QUADRATIC DIAMETER OF A METRIC SPACE AND ITS APPLICATION TO A PROBLEM IN ANALYSIS

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1. Introduction. The purpose of this note is to introduce the notion of quadratic diameter and to apply it to a problem in analysis concerning the measure of the set of values taken by a real-valued differentiable function on its critical set. It was proved by A. P. Morse [3] that if \( f(x) \) is a real-valued function defined on an open set of an \( n \)-dimensional Euclidean space which is \( n \)-times continuously partially differentiable, then the set of values taken by \( f(x) \) on its critical set (i.e. the set of all \( x \) at which all first partial derivatives of \( f(x) \) vanish) is a set of one-dimensional Lebesgue measure zero. We shall give a simple proof of this result of Morse for the case \( n = 2 \), under a weaker assumption that the first partial derivatives of \( f(x) \) satisfy the Lipschitz condition. (It is mentioned in [3] that his condition can be weakened and that it is sufficient to assume that all \( (n-1) \)th partial derivatives of \( f(x) \) are totally differentiable; but this weakened condition is still stronger than our condition for the case \( n = 2 \).) It is, however, to be observed that our method seems to work only for the case \( n = 2 \), and the problem of discussing the possibility of similar approach for the higher-dimensional cases \( n > 2 \) is still open.

2. Quadratic diameter. Let \( R = \{x, y, \cdots\} \) be a metric space with the metric \( d(x, y) \). For any subset \( S \) of \( R \) and for any two points \( x, y \) of \( S \), the quadratic distance \( d^{(2)}(x, y; S) \) of \( x, y \) with respect to \( S \) is defined by

\[
(2.1) \quad d^{(2)}(x, y, S) = \inf \sum_{i=1}^{\mu} (d(x_{i-1}, x_i))^2,
\]

where \( \inf \) is taken for all possible finite chains \( x_0 = x, x_1, \cdots, x_\mu = y \) from \( S \). (The length \( \mu \) of this chain is also arbitrary.) Further, the quadratic diameter \( \delta^{(2)}(S) \) of \( S \) is defined by

\[
(2.2) \quad \delta^{(2)}(S) = \sup_{x, y \in S} d^{(2)}(x, y; S).
\]

Finally, the quadratic variation \( v^{(2)}(S) \) of \( S \) is defined by

\[
(2.3) \quad v^{(2)}(S) = \inf \sum_{i=1}^{\mu} \delta^{(2)}(S_i),
\]

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where \( \inf \) is taken for all possible decompositions \( S = \bigcup_{i=1}^{p} S_i \) of \( S \) into a finite number of (not necessarily disjoint) closed subsets \( S_i, i=1, \ldots, p \).

We begin with simple observations concerning the quantities defined above.

**Lemma 2.1.**

\[
(2.4) \quad d^{(2)}(x, y; S) = d^{(2)}(y, x; S) \leq (d(x, y))^2,
\]

\[
(2.5) \quad d^{(2)}(x, z; S) \leq d^{(2)}(x, y; S) + d^{(2)}(y, z; S)
\]

for any \( x, y, z \in S \).

**Lemma 2.2.**

\[
(2.6) \quad \delta^{(2)}(S) \leq \delta^{(2)}(S) \leq (\delta(S))^2,
\]

where \( \delta(S) = \sup_{x, y \in S} d(x, y) \) is the ordinary diameter of \( S \). Further

\[
(2.7) \quad \delta^{(2)}(S_1 \cup S_2) \leq \delta^{(2)}(S_1) + \delta^{(2)}(S_2)
\]

if \( S_1 \) and \( S_2 \) are not disjoint, and

\[
(2.8) \quad \delta^{(2)}(S) = \delta^{(2)}(S)
\]

if \( S \) is connected.

The proofs of these lemmas are easy and so omitted.

Because of Lemma 2.1, \( d^{(2)}(x, y; S) \) may be considered as a quasi-metric on \( S \). It is possible that \( d^{(2)}(x, y; S) = 0 \) for two different points \( x, y \) of \( S \) (as will be shown in Lemma 2.3). In this case \( x, y \) are called \( d^{(2)} \)-equivalent. It is easy to see that this relation of \( d^{(2)} \)-equivalence is symmetric and transitive, and that for any \( x \in S \) the set \( E(x) \) of all \( y \in S \) with is \( d^{(2)} \)-equivalent with \( x \) is a closed subset of \( S \). Further, \( d^{(2)}(E(x), E(y)) = d^{(2)}(x, y; S) \) may be considered as a metric defined on the class of all equivalence classes \( E(x) \) of \( S \).

It is to be noticed that \( S_1 \subseteq S_2 \subseteq R \) and \( x, y \in S_1 \) imply

\[
(2.9) \quad d^{(2)}(x, y; S_1) = d^{(2)}(x, y; S_2),
\]

and that it is possible that the inequality holds in (2.9). Thus the quadratic diameter \( \delta^{(2)}(S) \) of \( S \) is not necessarily a monotone set function of \( S \).

We do not need the following Lemmas 2.3, 2.4 in the proof of our main result in §4. But a similar argument plays an important role in the proof of Lemma 4.3.

**Lemma 2.3.** If \( S \) is a subset of an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) (\( n \geq 1 \)), and if \( x, y \) are two points of \( S \) such that the segment \( [x, y] \)
connecting $x$ and $y$ is contained in $S$, then $d^{(2)}(x, y; S) = 0$.

**Proof.** For any integer $p \geq 1$, let $x_0 = x, x_1, \ldots, x_p = y$ be a chain of points from $S$ which divide the segment $[x, y]$ into $p$ equal parts. Then $\sum_{i=1}^{p} (d(x_{i-1}, x_i))^2 = (1/p)(d(x, y))^2$, and this implies that $d^{(2)}(x, y; S) = 0$.

**Corollary.** If $S$ is a convex subset of a Euclidean space $R^n$ $(n \geq 1)$, then $\delta^{(2)}(S) = 0$.

**Lemma 2.4.** If $x, y$ are two points of a subset $S$ of a metric space $R$ which can be connected by a rectifiable curve in $S$, then $d^{(2)}(x, y; S) = 0$.

**Proof.** Let $C = \{x_t | 0 \leq t \leq 1\}$ ($x_0 = x, x_1 = y$) be a rectifiable curve which connects $x$ and $y$ in $S$. Let $l$ be the curve length of $C$. Then, for any $\epsilon > 0$, there exist real numbers $t_0, t_1, \ldots, t_p$ such that $t_0 = 0 < t_1 < \ldots < t_p = 1, d(x_{t_i-1}, x_{t_i}) < \epsilon, i = 1, \ldots, p,$ and

$$
\sum_{i=1}^{p} d(x_{t_{i-1}}, x_{t_i}) < l + \epsilon.
$$

From this follows

$$
\sum_{i=1}^{p} (d(x_{t_{i-1}}, x_{t_i}))^2 < \epsilon(l + \epsilon)
$$

and this implies $d^{(2)}(x, y; S) = 0$.

**Corollary.** If $S$ is a subset of a metric space $R$ such that any two points $x, y$ of $S$ can be connected by a rectifiable curve in $S$, then $\delta^{(2)}(S) = 0$.

The question then arises to determine, given a metric space $R$, whether or not there exists a connected subset $S$ of $R$ with the property that $d^{(2)}(x, y; S) > 0$ for any two different points $x, y$ of $S$. We shall show in §3 that in a Hilbert space or in a Euclidean space $R^n$ with $n \geq 3$ there exists a simple curve $S = \{x_t | 0 \leq t \leq 1\}$ such that $d^{(2)}(x_s, x_t; S) > 0$ for any $s, t$ with $0 \leq s < t \leq 1$.

3. **Examples of continuous curves with a positive quadratic diameter.** Let $S = \{x_t | 0 \leq t \leq 1\}$ be a curve in a Hilbert space $R$ with the following properties: (i) the vectors $x_t - x_s$ and $x_{t'} - x_{s'}$ are orthogonal if $0 \leq s < t \leq s' < t'$, (ii) $d(x_s, x_t) = (t - s)^{1/2}$ if $0 \leq s < t \leq 1$. Then it is easy to see that

$$
\sum_{i=1}^{p} (d(x_{t_{i-1}}, x_{t_i}))^2 = (d(x_s, x_t))^2 = t - s
$$

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for any \( s, t_0, t_1, \ldots, t_p, t \) such that \( 0 \leq s = t_0 < t_1 < \cdots < t_p = t \leq 1 \).
From this follows that

\[
d^{(2)}(x_s, x_t; S) = t - s
\]

for any \( s, t \) with \( 0 \leq s < t \leq 1 \). An example of a curve \( S \) in a Hilbert space with the properties (i), (ii) can be given as follows: Let \( R = L^2(0, 1) \) be the \( L^2 \)-space of all real-valued square integrable functions \( x(u) \) defined on the unit interval \( I = \{ u \mid 0 \leq u \leq 1 \} \) with the usual norm

\[
\|x\| = \left( \int_0^1 |x(u)|^2 du \right)^{1/2} < \infty,
\]

and let \( S = \{ x_t \mid 0 \leq t \leq 1 \} \) be a curve in \( R \) defined by

\[
x_t(u) = \begin{cases} 
1, & 0 \leq u \leq t, \\
0, & t < u \leq 1.
\end{cases}
\]

Then it is easy to see that \( S \) has the properties (i), (ii) above. It is to be noticed that from Lemma 2.4 follows that \( S \) is not rectifiable. In fact it is easy to see that

\[
\sum_{i=1}^p d(x_t(i-1)/p, x_t/i/p) = p \cdot \frac{1}{p^{1/2}} = p^{1/2} \to \infty
\]
as \( p \to \infty \). Such curves have been discussed by N. Wiener [5], A. Kolmogoroff [1], and P. Lévy [2] in connection with the theory of Brownian motion.

In case \( R \) is a finite-dimensional Euclidean space \( R^n \) (\( n \geq 3 \)), the existence of a curve \( S = \{ x_t \mid 0 \leq t \leq 1 \} \) with the same property can be shown by using a Peano curve. In fact, let \( \{ (u_t, v_t) \mid 0 \leq t \leq 1 \} \) be the usual parametrization of the standard Peano curve filling the closed unit square \( Q = \{ (u, v) \mid 0 \leq u, v \leq 1 \} \) of \( R^2 \) with the property that, for any \( n = 1, 2, \ldots \) and for any \( i = 1, \ldots, 2^n \), the image of the closed interval

\[
I_i^{(n)} = \{ t \mid (i - 1)2^{-2n} \leq t \leq i \cdot 2^{-2n} \}
\]
is one of the \( 2^n \) closed squares

\[
Q_k,l^{(n)} = \{ (u, v) \mid (k - 1)2^{-n} \leq u \leq k2^{-n}, (l - 1)2^{-n} \leq v \leq l \cdot 2^{-n} \},
\]

\( k, l = 1, \ldots, 2^n \).

Then \( S = \{ x_t = (u_t, v_t, t) \mid 0 \leq t \leq 1 \} \) is a simple curve in \( R^3 \) whose
projection on the \((u,v)\)-plane fills up the closed unit interval \(\overline{Q} = \{(u,v) \mid 0 \leq u, v \leq 1\}\). It is easy to see that \(d^{(2)}(x_s, x_t; S) > 0\) for any \(s, t\) with \(0 \leq s < t \leq 1\).

On the other hand, it is impossible to construct similar examples in \(\mathbb{R}^2\). In fact, as will be shown in Theorem 2 below, \(\delta^{(2)}(S) = 0\) for any compact connected subset \(S\) of \(\mathbb{R}^2\) and hence \(d^{(2)}(x, y; S) = 0\) for any two points \(x, y\) of a continuous curve \(S\) in \(\mathbb{R}^2\).

If we consider the case when \(S\) is compact but not connected, then it is clear that \(\delta^{(2)}(S) > 0\). In fact, if \(S = S_1 \cup S_2\) is a decomposition of \(S\) into two disjoint compact subsets \(S_1\) and \(S_2\), then there exists a constant \(\eta > 0\) such that \(d(x, y) \geq \eta > 0\) for any \(x \in S_1\), \(y \in S_2\) and hence \(\delta^{(2)}(S) \geq d^{(2)}(x, y; S) \geq \eta^2 > 0\) for any \(x \in S_1\), \(y \in S_2\). Thus it seems as if there is nothing we can conclude about \(\delta^{(2)}(S)\) in general. But we can show that if \(S\) is a compact subset of \(\mathbb{R}^2\), then the quadratic variation \(v^2(S)\) of \(S\) is zero, or in other words, that, for any \(\eta > 0\), there exists a decomposition \(S = \bigcup_{i=1}^p S_i\) of \(S\) into a finite number of subsets \(S_i\), \(i = 1, \ldots, p\), such that \(\sum_{i=1}^p \delta^{(2)}(S_i) < \eta\). This will be proved in Theorem 1.

4. Quadratic diameter of a set in \(\mathbb{R}^2\).

**Lemma 4.1.** Let \(S\) be a compact subset of a two-dimensional Euclidean space \(\mathbb{R}^2\). If we denote by \(\alpha S\) a subset of \(\mathbb{R}^2\) which is homothetic with \(S\) with the ratio \(\alpha : 1\), then

\[
\delta^{(2)}(\alpha S) = \alpha^2 \delta^{(2)}(S).
\]

**Proof.** The proof is clear from definition.

**Lemma 4.2.** Let \(S\) be a compact subset of the closed unit square \(\overline{Q} = \{(u,v) \mid 0 \leq u, v \leq 1\}\) of \(\mathbb{R}^2\) with \(m^{(2)}(S) < \epsilon\), where \(m^{(2)}(S)\) denotes the two-dimensional Lebesgue measure of \(S\). Then there exists a decomposition \(S = \bigcup_{i=1}^p S_i\) of \(S\) into a finite number of closed subsets \(S_i\), \(i = 1, \ldots, p\), such that

\[
\sum_{i=1}^p \delta^{(2)}(S_i) < 2\epsilon.
\]

**Proof.** Since \(m^{(2)}(S) < \epsilon\), there exists an open set \(O\) of \(\mathbb{R}^2\) such that \(S \subseteq O\) and \(m^{(2)}(O) < \epsilon\). Since \(O\) is open, there exists a countable number of open squares \(Q_i, \ i = 1, 2, \ldots\), such that \(O = \bigcup_{i=1}^\infty Q_i\) and \(\sum_{i=1}^\infty m^{(2)}(Q_i) < \epsilon\). Since \(S\) is compact, there exists a positive integer \(p\) such that \(S \subseteq \bigcup_{i=1}^p Q_i\). If we put \(S_i = S \cap Q_i\) where \(\overline{Q_i}\) is the closure of \(Q_i, \ i = 1, \ldots, p\), then \(S = \bigcup_{i=1}^p S_i\) is a required decomposition. In fact, in the notation of (2.6), it is clear that \(\sum_{i=1}^p \delta^{(2)}(S_i)\)
Lemma 4.3. Let $S$ be a compact subset of the closed unit square $\overline{Q} = \{(u, v) \mid 0 \leq u, v \leq 1\}$ of $\mathbb{R}^2$ such that $m^2(\overline{Q} - S) < \varepsilon$, where $\varepsilon < \pi/16$ and $m^2(\overline{Q} - S)$ denotes the two-dimensional Lebesgue measure of $\overline{Q} - S$. Then
\begin{equation}
(4.3) \quad \delta^2(S) < 50 \varepsilon^{1/2}.
\end{equation}

Proof. Let $p$ be a positive integer such that $p^{-1} \leq \varepsilon^{1/2} < (p-1)^{-1}$. For any two points $x, y$ of $S$, let $x_0 = x, x_1, \cdots, x_p = y$ be the chain of points from $\overline{Q}$ of length $p$ which divides the segment $[x, y]$ in $p$ equal parts. We do not know whether these dividing points $x_1, \cdots, x_{p-1}$ belong to $S$ or not. But it is possible to show that for each $i = 1, \cdots, p-1$, there exists a point $x'_i$ from $S$ such that $d(x_i, x'_i) < 2(\varepsilon/\pi)^{1/2}$. In fact, if we denote by $K_i = K(x_i, 2(\varepsilon/\pi)^{1/2})$ the circle of radius $2(\varepsilon/\pi)^{1/2}$ ($<1/2$) whose center is at $x_i$, then at least one quadrant of this circle is contained in $\overline{Q}$. Since the area of $K_i$ is equal to $4\varepsilon$ and since $m^2(\overline{Q} - S) < \varepsilon$ by assumption, each $K_i$ must contain at least one point $x'_i$ which belongs to $S$, $i = 1, \cdots, p-1$.

Let us now put $x'_0 = x_0, x'_p = x_p$. Then $x'_0 = x, x'_1, \cdots, x'_p = y$ is a chain of points from $S$ which connects $x$ and $y$. Consequently,
\begin{align*}
d^{(2)}(x, y; S) &\leq \sum_{i=1}^{p} (d(x'_{i-1}, x'_i))^2 \\
&\leq \sum_{i=1}^{p} (d(x_{i-1}, x_i) + d(x_i, x'_i))^2 \\
&\leq p \left(2(\varepsilon/\pi)^{1/2} + \frac{1}{p} d(x, y) + 2(\varepsilon/\pi)^{1/2}\right)^2 \\
&\leq p \left(4(\varepsilon/\pi)^{1/2} + \frac{2^{1/2}}{p}\right)^2 \\
&\leq \left(1 + \frac{1}{\varepsilon^{1/2}}\right) \left(4(\varepsilon/\pi)^{1/2} + (2\varepsilon)^{1/2}\right)^2 \\
&\leq \varepsilon^{1/2}(1 + \varepsilon^{1/2}) \left(\frac{4}{\pi^{1/2}} + 2^{1/2}\right)^2 \leq 50\varepsilon^{1/2}.
\end{align*}

Theorem 1. If $S$ is a compact subset of a two-dimensional Euclidean space, then the quadratic variation of $S$ is zero, i.e. for any $\eta > 0$, there exists a decomposition $S = \bigcup_{i=1}^{p} S_i$ of $S$ into a finite number of closed subsets $S_i, i = 1, \cdots, p$, such that
Proof. Because of Lemma 4.1 it suffices to discuss the case when $S$ is a subset of the closed unit square $Q = \{(u, v) \mid 0 \leq u, v \leq 1\}$ of $\mathbb{R}^2$.

For any positive integer $n$, let us put

$$Q_{k,i}^{(n)} = \left\{ (u, v) \mid \frac{k-1}{2^n} < u < \frac{k}{2^n}, \frac{l-1}{2^n} < v < \frac{l}{2^n} \right\}$$

$k, l = 1, 2, \ldots, 2^n$, and

$$S_{k,i}^{(n)} = S \cap Q_{k,i}^{(n)}$$

where $Q_{k,i}^{(n)}$ is the closure of $Q_{k,i}^{(n)}$, $k, l = 1, \ldots, 2^n$. Then it is clear that

$$S = \bigcup_{k, l = 1}^{2^n} S_{k,i}^{(n)} = \bigcup_{(k, l) \in \Delta^{(n)}} S_{k,i}^{(n)}$$

where $\Delta^{(n)}$ denotes the set of all pairs of integers $(k, l)$, $k, l = 1, \ldots, 2^n$.

Let $\epsilon > 0$ be a positive number such that $4\epsilon + 50\epsilon^{1/2} < \eta$, and let us divide the set $\Delta^{(n)}$ into three classes $\Delta_1^{(n)}, \Delta_2^{(n)}, \Delta_3^{(n)}$, by the following conditions:

$$\Delta_1^{(n)} = \{(k, l) \mid m^{(2)}(S_{k,i}^{(n)}) < \epsilon m^{(2)}(Q_{k,i}^{(n)})\},$$

$$\Delta_2^{(n)} = \{(k, l) \mid \epsilon \cdot m^{(2)}(Q_{k,i}^{(n)}) \leq m^{(2)}(S_{k,i}^{(n)}) \leq (1 - \epsilon) m^{(2)}(Q_{k,i}^{(n)})\},$$

$$\Delta_3^{(n)} = \{(k, l) \mid (1 - \epsilon) m^{(2)}(Q_{k,i}^{(n)}) < m^{(2)}(S_{k,i}^{(n)})\}.$$ 

Thus, from the theory of Lebesgue measure in $\mathbb{R}^2$, it follows that, for any fixed $\epsilon > 0$, there exists an integer $n_0(\epsilon)$ such that

$$|\Delta_2^{(n)}| < \epsilon \cdot 2^{-2n}$$

for any $n > n_0(\epsilon)$, where $|\Delta_2^{(n)}|$ denotes the number of pairs of $(k, l)$ contained in $\Delta_2^{(n)}$.

From Lemma 4.2 it follows that, for any $(k, l) \in \Delta_1^{(n)}$, there exists a decomposition

$$S_{k,i}^{(n)} = \bigcup_{i=1}^{p_{k,i}} S_{k,i}^{(n)}$$

of $S_{k,i}^{(n)}$ into a finite number of closed subsets $S_{k,i}^{(n)}$, $i = 1, \ldots, p_{k,i}$, such that
Further, from Lemmas 4.1 and 4.3 follows that

\[(4.15) \quad \delta^{(2)}(S_{k,l}) < 50e^{1/2} \cdot 2^{-2n}\]

for any \((k, l) \in \Delta^{(n)}_3\). If we now consider the decomposition

\[(4.16) \quad S = \bigcup_{(k,l) \in \Delta^{(n)}_1} \bigcup_{i=1}^{2k} S^{(n)}_{k,l,i} \cup \bigcup_{(k,l) \in \Delta^{(n)}_2} S^{(n)}_{k,l},\]

then we see that

\[(4.17) \quad v^{(2)}(S) \leq \sum_{(k,l) \in \Delta^{(n)}_1} \sum_{i=1}^{2k} \delta^{(2)}(S^{(n)}_{k,l,i}) + \sum_{(k,l) \in \Delta^{(n)}_2} \delta^{(2)}(S^{(n)}_{k,l})\]

\[\leq |\Delta^{(n)}_1| \cdot 2e \cdot 2^{-2n} + |\Delta^{(n)}_2| \cdot 2 \cdot 2^{-2n}\]

\[+ |\Delta^{(n)}_3| \cdot 50e^{1/2} \cdot 2^{-2n},\]

where \(|\Delta^{(n)}_i|\) is the number of pairs \((k, l)\) contained in \(\Delta^{(n)}_i, i = 1, 2, 3\). Since \(|\Delta^{(n)}_1| \leq 2^{2n}, |\Delta^{(n)}_2| \leq 2^n\) for all \(n\) and \(|\Delta^{(n)}_3| < e^{-2n}\) for \(n > n_0(e)\), it follows that the right-hand side of (4.17) is smaller than \(2e + 2e + 50e^{1/2} < \eta\) if \(n > n_0(e)\). This proves Theorem 1.

**Theorem 2.** If \(S\) is a compact connected subset of \(R^n\), then \(\delta^{(2)}(S) = 0\).

**Proof.** This follows from Theorem 1 and the relation (2.8) of Lemma 2.2.

5. Functions satisfying the conditions of Lipschitz of type \((2, M)\).

Let \(R = \{x, y, \ldots\}\) be a metric space with a metric \(d(x, y)\). Let \(\alpha, M\) be two positive numbers. A real-valued function \(f(x)\) defined on a subset \(S\) of \(R\) is said to satisfy the condition of Lipschitz of type \((\alpha, M)\) on \(S\) if

\[(5.1) \quad |f(x) - f(y)| \leq M(d(x, y))^\alpha\]

for any \(x, y \in S\). We are mainly interested in the case \(\alpha = 1, 2\). (In case \(\alpha = 1\), our condition is reduced to the usual Lipschitz condition.)

For any real-valued function \(f(x)\) defined on \(S\), we denote by \(f(S)\) the set of values taken by \(f(x)\) on \(S\). The diameter \(\delta(f(S))\) of \(f(S)\)
is nothing but the oscillation \( \sup_{x \in S} f(x) - \inf_{x \in S} f(x) \) of \( f(x) \) on \( S \).

**Lemma 5.1.** Let \( f(x) \) be a real-valued function defined on a subset \( S \) of a metric space \( R \) which satisfies the condition of Lipschitz of type \((2, M)\) on \( S \). Then

\[
\delta(f(S)) \leq M \cdot \delta^{(2)}(S).
\]

**Proof.** It suffices to show that

\[
\left| f(x) - f(y) \right| \leq M \cdot d^{(2)}(x, y; S)
\]

for all \( x, y \in S \), and this follows immediately from the fact that

\[
| f(x) - f(y) | \leq \sum_{i=1}^{p} | f(x_{i-1}) - f(x_i) |
\]

\[
\leq M \sum_{i=1}^{p} (d(x_{i-1}, x_i))^2
\]

for any chain \( x_0 = x, x_1, \ldots, x_p = y \) from \( S \).

**Lemma 5.2.** Under the same assumptions as in Lemma 5.1, we have

\[
m^{(1)}(f(S)^-) \leq M \nu^{(2)}(S),
\]

where \( m^{(1)}(f(S)^-) \) denotes the one-dimensional Lebesgue measure of the closure \( f(S)^- \) of \( f(S) \). In particular, \( f(S)^- \) is a set of Lebesgue measure zero if \( \nu^{(2)}(S) = 0 \).

**Proof.** For any \( \epsilon > 0 \), let \( S = \bigcup_{i=1}^{p} S_i \) be a decomposition of \( S \) into a finite number of closed subsets \( S_i, i = 1, \ldots, p \), such that

\[
\sum_{i=1}^{p} \delta^{(2)}(S_i) < \nu^{(2)}(S) + \epsilon.
\]

It is then easy to see that

\[
m^{(1)}(f(S)^-) \leq \sum_{i=1}^{p} \delta(f(S_i))
\]

\[
\leq M \sum_{i=1}^{p} \delta^{(2)}(S_i)
\]

\[
< M \big( \nu^{(2)}(S) + \epsilon \big),
\]

which obviously implies (5.5) since \( \epsilon > 0 \) is arbitrary.

**Theorem 3.** Let \( f(x) \) be a real-valued function defined on a compact subset \( S \) of \( R^2 \) which satisfies the condition of Lipschitz of type \((2, M)\) on
S. Then the set \( f(S) \) of values taken by \( f(x) \) on \( S \) is a set of one-dimensional Lebesgue measure zero. Further, \( f(x) \) is constant on every component of \( S \). In particular, \( f(x) \) is a constant if \( S \) is connected.

**Proof.** The proof follows immediately from Theorems 1, 2 and Lemmas 5.1, 5.2.

It is easy to see that, for any subset \( S \) of a metric space \( R \) with \( \delta^{(2)}(S) > 0 \), there exists a nonconstant function \( f(x) \) defined on it which satisfies the condition of Lipschitz of type (2.1) on \( S \). In fact, for any fixed \( x_0 \in S \), the function \( f(x) = d^{(2)}(x, x_0; S) \) is a nonconstant function defined on \( S \) and satisfies

\[
|f(x) - f(y)| = |d^{(2)}(x, x_0; S) - d^{(2)}(y, x_0; S)| 
\leq d^{(2)}(x, y; S) \leq (d(x, y))^2.
\]

In particular, if \( S = \{x_t | 0 \leq t \leq 1\} \) is a curve in a Hilbert space or a Euclidean space \( R^n (n \geq 3) \) discussed in §3 with the property that \( \delta^{(2)}(x_s, x_t; S) > 0 \) for any \( s, t \) with \( 0 \leq s < t \leq 1 \), then \( f(x_t) = t \) is a nonconstant function which satisfies the condition of Lipschitz of type (2.1) on \( S \).

6. Application to a problem in analysis. Let \( Q = \{(u, v) | 0 < u, v < 1\} \) be the open unit square in \( R^2 \), and let \( f(x) = f(u, v) \) be a real-valued function defined on \( Q \). \( f(x) = f(u, v) \) is said to belong to the class \( C^I \) if all first partial derivatives of \( f(u, v) \) exist and are continuous on \( Q \). \( f(x) = f(u, v) \) is said to belong to the class \( C^I(\alpha, M) \) if further all first partial derivatives of \( f(y, v) \) satisfy the Lipschitz condition of type \( (\alpha, M) \). We are mainly interested in the case \( \alpha = 1 \).

Let \( f(x) = f(u, v) \) be a function belonging to the class \( C^I \). A point \( x = (u, v) \in Q \) is called a critical point of \( f \) if all first partial derivatives vanish at \( x = (u, v) \). The set \( S(f) \) of all critical points of \( f \) is called the critical set of \( f \). It is clear that \( S(f) \) is a relatively closed subset of \( Q \), and hence is an \( F^\alpha \)-set.

**Lemma 6.1.** If \( f(x) \) belongs to \( C^I(1, M) \), then \( f(x) \) satisfies the condition of Lipschitz of type \((2, M)\) on its critical set.

**Proof.** The proof is easy and so is omitted.

**Theorem 4.** Let \( f(x) = f(u, v) \) be a real-valued function defined on the open unit square \( Q = \{(u, v) | 0 < u, v < 1\} \) of \( R^2 \) which belongs to the class \( C^I(1, M) \). Then the set \( f(S(f)) \) of values taken by \( f(x) \) on its critical set \( S(f) \) is a set of one-dimensional Lebesgue measure zero. Further, \( f(x) \) is constant on every component of \( S(f) \).

**Proof.** The proof follows easily from Theorem 3 and Lemma 6.1.
Remark. It was shown by H. Whitney [4] that there exists a function \( f(x) = f(u, v) \) defined on the open unit square \( Q \) of \( R^2 \) with continuous first partial derivatives which is not constant on a connected part of its critical set \( S(f) \). This fact shows that the continuity of the first partial derivatives is not sufficient for Theorem 4, and that something like the condition of Lipschitz of type \((1, M)\) for the first partial derivatives of \( f(u, v) \) is necessary for Theorem 4.

If, on the other hand, all second partial derivatives of \( f(u, v) \) exist and are continuous, then the proof of Theorem 4 becomes much easier. In fact, in this case it is easy to see that, for any \( x \in S(f) \), there exists a neighborhood \( V(x) \) of \( x \) such that \( f(x) \) satisfies the condition of Lipschitz of type \((2, \epsilon)\) on \( S(f) \cap V(x) \). From this follows that, for any \( \epsilon > 0 \), there exists a sequence of squares \( Q_i, i = 1, 2, \ldots \), such that (i) \( \bigcup_{i=1}^{\infty} Q_i = Q \), (ii) \( \sum_{i=1}^{\infty} m^{(2)}(Q_i) < m^{(2)}(Q) + \epsilon = 1 + \epsilon \), and (iii) \( f(x) \) satisfies the condition of Lipschitz of type \((2, \epsilon)\) on each \( S_i = S(f) \cap Q_i, i = 1, 2, \ldots \). From Lemma 6.1 then follows

\[
m^{(1)}(f(S)) \leq \sum_{i=1}^{\infty} m^{(1)}(S_i)
\]

\[
\leq \sum_{i=1}^{\infty} \delta(f(S_i)) \leq \epsilon \sum_{i=1}^{\infty} \delta^{(2)}(S_i)
\]

\[
\leq \epsilon \sum_{i=1}^{\infty} \delta^{(2)}(Q_i) \leq 2\epsilon \sum_{i=1}^{\infty} m^{(2)}(Q_i)
\]

\[
< 2\epsilon(1 + \epsilon).
\]

Bibliography


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