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**TWO NOTES ON NILPOTENT GROUPS**

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I

We extend a theorem of Rédei and Szép.¹ Our proof is quite straightforward, and employs a method of considerably more general applicability.²

The *lower central series* of a group $G$ is formed by taking $G_1 = G$, and successively defining $G_{n+1}$ to be the commutator $(G_n, G)$. $G$ is *nilpotent* if some $G_{N+1} = 1$. If $A$ and $B$ are subgroups of $G$, $A \vee B$ is the subgroup generated by the elements of $A$ and of $B$ together, and $A^m$ the subgroup generated by the $m$th powers of elements of $A$.

**Theorem.** Let $A$ and $K$ be subgroups of a nilpotent group $G$, and let $A^m = 1$ for some integer $m$. Then, for any $n \geq 1$,

$$(A \vee K)^n = (A^m \vee K)^n \implies (A \vee K)^n = K_n.$$  

We may clearly suppose that $G = A \vee K$. The elements of $G_r$ can be written as products of commutators of order $r$:

$$(x_1, \cdots, x_r) = (((x_1, x_2), x_3) \cdots, x_{r-1}), x_r).$$

Let $C_r$ be the subgroup generated by those commutators for which

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¹ L. Rédei and J. Szép, Monatshfte für Mathematik vol. 55, p. 200. The present proof avoids "counting arguments" and the attendant finiteness conditions; for $n = 1$ the present argument reduces substantially to that of Rédei and Szép. We remark that the hypothesis $A^m = 1$ admits various modifications.

some $x_i$ is in $A$, and $D$, by those for which some $x_i$ is in $A^m$. From the identity

$$(x, yz) = (x, y)(x, z)(z, x, y)$$

it follows that all commutators are linear in the $x_i$, modulo commutators of higher order. In particular, it follows that

$$(A \cup K)_n = K_n \cup C_n,$$

$$(A^m \cup K)_n = K_n \cup D_n,$$

$$D_n \subseteq C_n^m \cup C_{n+1},$$

(1)

and, since $A^{m^*} = 1$, that

$$C_n^{m^*} \subseteq C_{n+1}.$$  

(2)

From the hypothesis that $(A \cup K)_n = (A^m \cup K)_n$, hence that $K_n \cup C_n = K_n \cup D_n$, we have $C_n \subseteq K_n \cup D_n$ and, from (1),

$$C_n \subseteq K_n \cup C_n^m \cup C_{n+1}.$$  

(3)

By the evident rule $(L \cup M)_n \subseteq L^m \cup M^m \cup (L, M)$, from

$$C_n \subseteq K_n \cup C_n^m \cup C_{n+1}$$

we deduce that

$$C_n^m \subseteq K_n^m \cup C_n^m \cup C_{n+1} \cup G_{2n},$$

$$C_n \subseteq K_n \cup C_n \cup C_{n+1},$$

and, by (3), that

$$C_n \subseteq K_n \cup C_n^{m^*} \cup C_{n+1}.$$  

Applying this argument $e-1$ times to (3) gives

$$C_n \subseteq K_n \cup C_n^{m^*} \cup C_{n+1},$$

whence, by (2),

$$C_n \subseteq K_n \cup C_{n+1}.$$  

(4)

From the Lie-Jacobi congruences

$$(x, y)(y, x) = 1, \quad (x, y, z)(y, z, x)(z, x, y) \equiv 1 \pmod{G_n},$$

it follows that every $(x_1, \ldots, x_{k+1})$ with $x_{k+1}$ in $A$ is expressible, modulo $G_{k+2}$, as a product of such factors with $x_i$ in $A$ for some $i \leq k$: in short,
Assuming now

\[ C_k \subset K_k \vee C_{k+1} \]

and substituting, we obtain

\[ (C_k, K) \subset K_{k+1} \vee C_{k+2}, \]

\[ (C_k, A) \subset (K_k, A) \vee C_{k+2} \subset (C_k, K) \vee C_{k+2} \subset (C_k, K), \]

whence

\[ C_{k+1} \subset K_{k+1} \vee C_{k+2}. \]

By iteration, it follows from (4) that

\[ C_n \subset K_n \vee K_{n+1} \vee \cdots \vee K_n \vee C_{n+1} \subset K_n \vee C_{n+1}. \]

Since \( G_{n+1} = 1 \) by hypothesis,

\[ C_n \subset K_n, \]

whence \( K_n \vee C_n = K_n \) and \( (A \vee K)_n = K_n \), as required.

II

By a uniform method\(^3\) we establish easily two results that are fairly obvious from well known considerations, and a further result (Theorem 2.1) which answers for nilpotent groups a question regarding identical relations in groups that was raised by B. H. Neumann.\(^4\)

We employ standard notation for commutators: \((x_1, \cdots, x_n) = (\cdots ((x_1, x_2), x_3), \cdots, x_n)\), and for the lower central series: \(G = G_1, G_{n+1} = (G_n, G)\).

**Lemma 1.** Let \( F \) be a finitely generated free group, and \( R \) a normal subgroup of \( F \). Then, for each \( n \geq 1 \), \( R = [S_n, R_{n+1}] \), the normal subgroup generated by a finite set \( S_n \) together with \( R_{n+1} = F/F_{n+1} \).

**Proof.** Proceed inductively from the vacuous case \( n = 0 \). Since \( F/F_{n+1} \) is a finitely generated abelian group, so is its subgroup \( R_{n+1}/R_{n+2} \). Let \( T = \{ r_i \} \) be a finite set of elements of \( R_{n+1} \) such that the cosets \( r_i R_{n+2} \) generate \( R_{n+1}/R_{n+2} \). Evidently \( R = [S_n, R_{n+1}] \) implies \( R = [S_n, T, R_{n+2}] \).

**Theorem 1.1.** Every finitely generated nilpotent group is definable by a finite set of relations.

\(^3\) For the method, see references given in footnote 2.

Proof. If \( G = F/R \) is nilpotent, say \( G_{n+1} = 1 \), we have \( R_{n+1} = F_{n+1} = [(x_1, \ldots, x_{n+1})] \), all sets \( x_1, \ldots, x_n \) of generators for \( F \). Hence \( R = [S_n, R_{n+1}] \) is defined by a finite set of relations.

Theorem 1.2. In a finitely generated group which is known to be nilpotent the word-problem is decidable.

Proof. Let \( G = F/R \) and \( G_{n+1} = 1 \). Suppose we have an expression for the word \( w \) in the form \( w = r_{n-1}w_n \), where \( r_{n-1} \) is in \( R \) and \( w_n \) is in \( F_n \). By reference to the finitely generated abelian group \( F_n/F_{n+1} \), we can obtain an expression \( w_n = r_nw_{n+1} \), \( r \) in \( R \), \( w_{n+1} \) in \( F_{n+1} \), if any such exists. Proceeding thus, either \( w = r_1r_2 \cdots r_{n-1}w_n \) where \( w_n \) is not in \( [R, F_{n+1}] \) and hence \( w \) is not in \( R \).

A normal subgroup \( W \) of the free group \( F \) is a word group if it is defined by certain words \( w(\xi_1, \ldots, \xi_n) \) under all substitutions of elements of \( F \) for the \( \xi_i \). For any group \( G \), let \( F \) be a denumerably generated free group; the group \( W_G \) of identical relations for \( G \) is the normal subgroup of \( F \) defined by all words \( w(\xi_1, \ldots, \xi_n) \) that equal 1 under all substitutions of elements of \( G \) for the \( \xi_i \).

Lemma 2. Let \( F \) be a free group and \( W \) a word subgroup of \( F \). Then, for each \( n \geq 1 \), \( W = \{S_n, W_{n+1}\} \), the word group defined by a finite set \( S_n \) of words, in at most \( n \) indeterminates, together with \( W_{n+1} = W \cap F_{n+1} \).

Proof. Induction as for Lemma 1. Consider the set of all relations of the form

\[
(1) \quad r = \prod c_i \cdot s
\]

where the \( c_i \) are commutators of generators of \( F \) of order \( n+1 \), \( \prod c_i \neq 1 \), and \( s \) is in \( F_{n+2} \). Each \( c_i \) contains at most \( n+1 \) generators. Let \( X \) be the set of generators occurring in some \( c_{i_0} \) in \( r \). Substituting \( x_i \rightarrow 1 \) for all generators \( x_i \) not in \( X \), we derive from \( r \) a relation

\[
(2) \quad r' = \prod' c_i \cdot s'
\]

where \( \prod' c_i \) is a partial product of that occurring in (1) and contains

\footnote{It is understood that \( G \) is defined by a finite set of relations, whence a finite set for \( F_n \) modulo \( F_{n+1} \) can be obtained, say, by a simplification of the Reidemeister-Schreier process. It suffices for Theorem 2.1, in fact, to assume that the \( G_n \) have intersection 1. To see this, test, for \( n = 1, 2, \ldots \), the two conditions: (i) \( w \) is in \( [R, F_n] \); and (ii) \( w \) is not equal, in \( F \), to any product \( \prod u_i r_i u_i^{-1} \) where the \( u_i \) and \( r_i \) together are of total length less than \( n \). For some finite \( n \) either (i) must fail and so \( w \) is not in \( R \), or (ii) must fail, whence \( w \) is in \( R \).}
at least the factor $c_{1p}$. Therefore, in

\[(3) \quad r'' = r \cdot r'^{-1} = \prod c_{i} \cdot s''\]

the product contains fewer factors than that in (1). If we repeat this construction, each relation (1) is obtained as a consequence of relations (2), each involving at most $n+1$ generators. Now, all the relations (2) are equivalent, for the purpose of defining $W$, to the corresponding relations (2'') in the generators $x_1, \ldots, x_{n+1}$, and, by Lemma 1, these possess a finite basis $T$ modulo $W_{n+2}$. Thus, if $W = \{S_n, W_{n+1}\}$, then $W = \{S_n, T, W_{n+2}\}$.

**Theorem 2.1.** A nilpotent group $G$ possesses a finite basis of identical relations.

**Proof.** If $G_{n+1}=1$, then $F_{n+1} \subseteq W_0 \subseteq F$. By Lemma 2, $W = \{S_n, W_{n+1}\}$ where $S_n$ is finite. But $W_{n+1} = F_{n+1}$ is defined by the single word $(\xi_1, \ldots, \xi_{n+1})$, whence $W$ has a finite basis. We note that, by multiplying together all the words in this basis, taken with distinct indeterminates, we obtain a single word which constitutes a basis for $W$.

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