A GENERALIZATION OF THE RIEMANN INTEGRAL

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By considering certain kinds of sets as exceptional in the definition of the Riemann integral, a variety of possible generalizations of the integral concept for bounded functions is obtained, one of which is the Lebesgue integral. Such an extension has been discussed by E. H. Hanson [1], whose class of exceptional sets, however, does not include the Lebesgue integral as a member of the associated class of integrals.

We consider a property $P$ of sets relative to intervals, which will always be closed on the left and open on the right. We write $(S, I)^P$ or $(S, I)^{P^c}$ according as the set $S$ has the property $P$ or does not have the property $P$ relative to the interval $I$, and suppose that $P$ is such that

(a) if $(S, I)^P$ and $T \subseteq S$, then $(T, I)^P$,
(b) if $(S, I)^P$, then $(CS, I)^{P^c}$, where $CS$ is the complement of $S$.

We associate two interval functions with every admissible property $P$ and bounded real function $f(x)$, defined on the interval $[0, 1)$, as follows:

$$\phi(P, f; I) = \inf \left\{ y \mid (E(f(x) > y), I)^P \right\}$$
and

$$\psi(P, f; I) = \sup \left\{ y \mid (E(f(x) < y), I)^P \right\},$$

where $E(f(x) > y)$ and $E(f(x) < y)$ are the sets of points for which $f(x) > y$ and $f(x) < y$, respectively.

We define the upper and lower $P$-integrals of $f(x)$ as the upper Burkill integral [2] of $\phi(P, f; I)$ and the lower Burkill integral of $\psi(P, f; I)$, respectively; i.e.,

$$P \int_a^b f(x) \, dx = B \int_a^b \phi(P, f; I)$$
and

$$P \int_{-\infty}^{\infty} f(x) \, dx = B \int_{-\infty}^{\infty} \psi(P, f; I).$$

If the upper and lower $P$-integrals of $f(x)$ are equal, we say that the $P$-integral, $P \int f(x) \, dx$, exists, and we have

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\[ P \int f(x) \, dx = \int_{\mathcal{P}} f(x) \, dx = \int_{\mathcal{P}} f(x) \, dx. \]

We note in passing that the P-integral of \( f(x) \) exists if and only if the Burkill integrals of \( \phi(P, f; I) \) and \( \psi(P, f; I) \) both exist and are equal.

For the case where \((S, I)^P\) if the set \( S \cap I \) is empty, the P-integral is the Riemann integral and, as we shall show, for the case where \((S, I)^P\) if the relative exterior measure of \( S \) in \( I \) is less than 1/2, the P-integral is the Lebesgue integral for bounded functions. Hanson considered those properties \( P \) for which \((S, I)^P\) if \( S \cap I \) belongs to a class \( \mathcal{E} \) of sets such that

1. if \( E_1 \in \mathcal{E} \) and \( E_2 \subseteq E_1 \), then \( E_2 \in \mathcal{E} \),
2. if \( E_n \in \mathcal{E} \), \( n = 1, 2, \ldots \), and \( E = \bigcup_{n=1}^{\infty} E_n \),
3. if \( E \) is an interval, then \( E \in \mathcal{E} \).

It is clear that every such property obeys our conditions (a) and (b).

To see that no P-integral is the Lebesgue integral if it is obtained from a class \( \mathcal{E} \) of sets satisfying the above conditions (a), (b), and (γ), we first note that in order to yield the Lebesgue integral the class \( \mathcal{E} \) must contain only sets of measure zero. For, if \( \mathcal{E} \) contains a set \( S \) whose exterior measure is positive, then the characteristic function of \( S \) has Lebesgue integral different from zero and P-integral equal to zero. Now, suppose \( \mathcal{E} \) contains only sets of measure zero. Let \( S \) be any measurable set such that the measures of the intersections of \( S \) and \( CS \) with \( I \) are both positive for every interval \( I \subseteq [0, 1) \).

Then the P-integral associated with the class \( \mathcal{E} \) of sets does not exist for the characteristic function of \( S \), but the Lebesgue integral of this function does exist.

There are properties \( P \), which satisfy our conditions, for which the P-integral is not additive. For example, let \((S, I)^P\) if \( S \cap I \) is a proper subset of the rational numbers in \( I \). Let \( R \) be the set of rational numbers in \([0, 1)\) and let \( Q \) be the set of those numbers in \( R \) which are of the form \( k/2^n \), where \( k \) and \( n \) are positive integers. Then, let

\[
f(x) = \begin{cases} 
1 & \text{if } x \in Q, \\
0 & \text{if } x \notin Q
\end{cases}
\]

and

\[
g(x) = \begin{cases} 
1 & \text{if } x \in R - Q, \\
0 & \text{if } x \notin R - Q.
\end{cases}
\]

Then
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\[ f(x) + g(x) = \begin{cases} 1 & \text{if } x \in R, \\ 0 & \text{if } x \notin R. \end{cases} \]

We may readily verify that \( P \int f(x) \, dx = 0 \), \( P \int g(x) \, dx = 0 \), and \( P \int \ast (f(x) + g(x)) \, dx = 1 \).

However, the P-integral is additive if \( P \) is such that \( (S, I)^P \) and \( (T, I)^P \) imply \( (S \cup T, I)^P \). Indeed, we have the following theorem.

**Theorem 1.** If \( P \) has an associated \( P' \) such that \( (S, I)^{P'} \) and \( (T, I)^{P'} \) imply \( (S \cup T, I)^{P'} \), and the existence of the \( P \)-integral of a bounded function implies the existence of its \( P' \)-integral, then the existence of the \( P \)-integrals of \( f(x) \) and \( g(x) \) implies the existence of the \( P' \)-integral of \( f(x) + g(x) \) and the equality

\[ P \int (f(x) + g(x)) \, dx = P \int f(x) \, dx + P \int g(x) \, dx. \]

**Proof.** It is an almost immediate consequence of the hypothesis that the \( P' \)-integral of a bounded function \( f(x) \) exists and is equal to the \( P \)-integral of \( f(x) \) if and only if the \( P \)-integral of \( f(x) \) exists. Now, let \( \epsilon > 0 \). There is an \( \eta > 0 \) such that for every partition of \([0, 1)\) into intervals \( I_1, I_2, \ldots, I_n \) whose lengths are less than \( \eta \), there are real numbers \( y_1, y_2, \ldots, y_n; z_1, z_2, \ldots, z_n \) such that

\[ (E(f(x) > y_k), I_k)^{P'} \quad \text{and} \quad (E(g(x) > z_k), I_k)^{P'} \]

for every \( k = 1, 2, \ldots, n \), and

\[ \sum_{k=1}^n y_k l(I_k) < P' \int f(x) \, dx + \frac{\epsilon}{2}, \quad \sum_{k=1}^n z_k l(I_k) < P' \int g(x) \, dx + \frac{\epsilon}{2}, \]

where \( l(I_k) \) is the length of \( I_k \). Now,

\[ E(f(x) + g(x) > y_k + z_k) \subset E(f(x) > y_k) \cup E(g(x) > z_k) \]

so that

\[ (E(f(x) + g(x) > y_k + z_k), I_k)^P. \]

Hence,

\[ \sum_{k=1}^n \phi(P, f + g; I_k) \leq \sum_{k=1}^n (y_k + z_k)l(I_k) = \sum_{k=1}^n y_k l(I_k) + \sum_{k=1}^n z_k l(I_k) \]

\[ < P \int f(x) \, dx + P \int g(x) \, dx + \epsilon, \]

since \( P \int f(x) \, dx = P' \int f(x) \, dx \) and \( P \int g(x) \, dx = P' \int g(x) \, dx \). As this holds
for every partition of \([0, 1)\) into intervals whose lengths are less than \(\eta\), it follows that
\[
P \int f(x) + g(x) \, dx < P \int f(x) \, dx + P \int g(x) \, dx + \epsilon.
\]
Similarly,
\[
P \int f(x) + g(x) \, dx > P \int f(x) \, dx + P \int g(x) \, dx - \epsilon.
\]
Since \(\epsilon > 0\) is arbitrary, \(P \int (f(x) + g(x)) \, dx\) exists, and
\[
P \int (f(x) + g(x)) \, dx = P \int f(x) \, dx + P \int g(x) \, dx.
\]

**Corollary.** The P-integral is additive if \(P\) is such that \((S, I)^P\) and \((T, I)^P\) imply \((S \cup T, I)^P\).

**Proof.** The conditions of Theorem 1 are satisfied with \(P = P'\).

We now show that the examples of properties \(P\) given above yield the Riemann and Lebesgue integrals for bounded functions.

**Theorem 2.** If \((S, I)^P\) means that \(S \cap I\) is empty, then the P-integral of \(f(x)\) agrees with its Riemann integral.

The proof is clear.

**Theorem 3.** If \((S, I)^P\) means that \(S\) has relative exterior measure less than \(1/2\) in \(I\), then the P-integral of a bounded function \(f(x)\) exists if and only if its Lebesgue integral exists, and then the two integrals are equal.

**Proof.** By a theorem of Kamke [3], \(f(x)\) is measurable, hence integrable, since it is bounded, if and only if it is approximately continuous almost everywhere.

Suppose \(f(x)\) is approximately continuous almost everywhere. Then, for every \(\epsilon > 0\), for every \(x\), except for those belonging to a certain set of measure zero, for every interval \(I\) containing \(x\) which is sufficiently small, the set of points \(y\) for which \(f(y) > f(x) + \epsilon\) and \(f(y) < f(x) - \epsilon\) are both of relative exterior measure less than \(1/2\) in the interval \(I\). A routine argument now shows that the P-integral of \(f(x)\) exists and is equal to its Lebesgue integral.

If \(f(x)\) is nonmeasurable, then since the set of points of approximate continuity of any function is measurable, the set of points of approximate discontinuity of \(f(x)\) is measurable and of positive
measure. It follows that there are real numbers $r$ and $s$, with $r > s$, such that the sets $E(f(x) > r)$ and $E(f(x) < s)$ both have positive upper exterior metric density at all the points of a set of positive measure. By the Lebesgue Density Theorem, these sets both have exterior metric density which exists and is equal to one at all the points of a set of measure $k > 0$. It readily follows that

$$P \int^* f(x) \, dx - P \int_* f(x) \, dx > k(r - s) > 0,$$

so that the P-integral of $f(x)$ does not exist.

The same proof shows that for the property $P'$, where $(S, I)^{P'}$ means that the relative exterior measure of $S$ is less than $1/4$ in $I$, the $P'$-integral is also the Lebesgue integral. Hence, the property $P$ of Theorem 3 satisfies the conditions of Theorem 1.

Our purpose here was to introduce the P-integral and show that there is a property $P$ for which the P-integral is the Lebesgue integral. We hope to give a detailed discussion of the partial order of abstract P-integrals elsewhere.

REFERENCES

1. E. H. Hanson, The $\tau$-limit, Ph.D. Dissertation, The Ohio State University, 1934.


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