ON A CONJECTURE CONCERNING DOUBLY STOCHASTIC MATRICES

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A doubly stochastic (d.s.) matrix is a real $n \times n$ matrix $P=(p_{ij})$ such that

\begin{align}
(1) & \quad p_{ij} \geq 0, \quad 1 \leq i \leq n, 1 \leq j \leq n, \\
(2) & \quad \sum_i p_{ij} = 1, \quad 1 \leq i \leq n,
\end{align}

and

\begin{align}
(3) & \quad \sum_i p_{ij} = 1, \quad 1 \leq j \leq n.
\end{align}

We introduce a partial order among d.s. matrices by defining

\begin{align}
(4) & \quad P_1 < P_2 \\
& \quad \text{to mean there exists a d.s. matrix } P_2 \text{ such that } \\
(5) & \quad P_1 = P_2 P_2.
\end{align}

We introduce a partial order among real vectors $a = (a_1, \ldots, a_n)$ of our real $n$-dimensional space $E$ by defining

\begin{align}
(6) & \quad a < b \quad \text{to mean for each real convex } \phi \\
& \quad \sum_i \phi(a_i) \leq \sum_i \phi(b_i).
\end{align}

By [HLP, p. 89], $a < b$ if and only if there exists a d.s. matrix $P$ such that

\begin{align}
& \quad a = Pb = \left( \sum_i p_{1i} b_i, \ldots, \sum_i p_{ni} b_i \right).
\end{align}

This implies that for each real $n$-vector $a$,

\begin{align}
(8) & \quad P_1 < P_2 \Rightarrow P_1 a < P_2 a.
\end{align}

Kakutani has raised the following conjecture.

CONJECTURE. If, for each real $n$-vector $a$, $P_1 a < P_2 a$, then $P_1 < P_2$. By [HLP, p. 89] if the hypothesis is satisfied there exists a d.s.

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matrix \( P_3^a \) such that

\[
P_1^a = P_3^a P_3^a.
\]

The issue is to show that if there exists such a \( P_3^a \) for each vector \( a \), there exists a d.s. \( P_3 \) independent of \( a \) such that \( P_1 = P_3 P_3 \).

Let \( \mathcal{P} \) be the collection of vectors with non-negative components. Requirement (1) is equivalent to the requirement

\[
P \mathcal{P} \subseteq \mathcal{P}.
\]

If \( e \) is the vector all of whose components are unity, then requirement (2) is equivalent to requirement

\[
P e = e,
\]

i.e., \( e \) is a characteristic vector of characteristic value unity. If \( \hat{e} \) is the element of \( \overline{E} \), the conjugate space of \( E \), whose value at \( a \in E \) is given by \( (\hat{e}, a) = \sum a_i \), then \( \hat{e}_1 = \{ a \mid (\hat{e}, a) = 0 \} \) is the set of vectors \( a \in E \) whose components add up to zero. Requirement (3) is equivalent to

\[
P(\hat{e}_1) \subseteq \hat{e}_1.
\]

Proof of conjecture. Suppose now \( P_1^a < P_3^a \) for each \( a \in E \). Consider a mapping \( \psi : P_3^E \to P_1^E \) defined as follows: if \( b \in P_3^E \), for some \( a \in E \) we have \( b = P_3^a \), then let \( \psi(b) = P_1^a \). We first prove (i) that we have a valid definition, i.e., \( \psi(b) \) is uniquely defined by the above and then we prove (ii) that \( \psi : P_3^E \to P_1^E \) is a linear transformation.

In order to prove (i) suppose that \( P_3^a' = P_3^a'' = b \). We wish to show that \( P_1^a' = P_1^a'' \). If \( P_3^a' = P_3^a'' \), then \( P_3^a' - a'' = 0 \), the zero vector. Since \( P_1^a' - a'' < P_3^a' - a'' \), by (9) we deduce that \( P_1^a' - a'' = 0 \) and \( P_1^a' = P_1^a'' \). Thus we have shown that \( \psi \) is uniquely defined.

The linearity (ii) of \( \psi \) is now trivial. If \( \alpha \) is a real scalar and \( P_3^a = b \), then \( P_3^\alpha a = \alpha b \) and \( \psi(ab) = P_1^\alpha a = \alpha P_1^a = \alpha \psi(b) \). Also if \( P_3^a' = b' \) and \( P_3^a'' = b'' \), then \( P_1^a' + a'' = b' + b'' \) and \( \psi(b' + b'') = P_1^a(a' + a'') = P_1^a' + P_1^a'' = \psi(b') + \psi(b''). \)

Suppose \( P_3^a \in \mathcal{P} \). Since \( P_1^a < P_3^a \) by (9) there exists a d.s. \( P_3 \) such that \( P_3^a = P_2^a P_3^a \in \mathcal{P} \). Thus \( \psi(P_3 \cap P_3^E) \subseteq \mathcal{P} \). Since \( P_1^a \) and \( P_3^a \) are d.s. matrices, \( P_1^e = P_3^e = e \) and so \( \psi(e) = e \). If by \( (P_3^a) \), we denote the \( i \)th component of \( P_3^a \), then by \([\text{HLP, p. 89}]
\) and the assumption \( P_1^a < P_3^a \) we have

\[
\sum_i (P_1^a)_i = \sum_i (P_3^a)_i.
\]
In particular
\[ \psi(\bar{e}^1 \cap P^3E) \subseteq \bar{e}^1. \]

We can now extend \( \psi \) to a function \( \Psi \) on all of \( E \) by letting \( \Psi(b) = 0 \) for each \( b \in E \cap (P^3E)' \), the complement of \( P^3E \), and \( \Psi(b) = \psi(b) \) for each \( b \in P^3E \). Now \( \Psi \) is a linear transformation satisfying the requirements

\[ \Psi(P) \subseteq P, \quad \Psi(e) = e, \quad \Psi(\bar{e}^1) \subseteq \bar{e}^1. \]

Therefore we can represent \( \Psi \) by a d.s. matrix \( P^2 \). We now have \( P^1a = P^2P^3a \) for each \( a \in E \) and therefore \( P^1 = P^2P^3 \), thus establishing the conjecture.

It can readily be shown that if \( P^1 < P^3 \), then for each \( j = 1, 2, \ldots, n \)

\[ (p_{1j}^1, p_{2j}^1, \ldots, p_{nj}^1) < (p_{1j}^3, \ldots, p_{nj}^3). \]

It would be interesting to establish the converse.

Reference


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