A COMBINATORIAL PROBLEM ON ABELIAN GROUPS

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1. Introduction. Suppose we are given a finite abelian group $A$ of order $n$, the group operation being addition. If

$$(a_1, a_2, \ldots, a_n)$$

$$(c_1, c_2, \ldots, c_n)$$

is a permutation of the elements of $A$, then the differences $c_i - a_i = b_1, \ldots, c_n - a_n = b_n$ are $n$ elements of $A$, not in general distinct, such that $\sum_{i=1}^{n-1} b_i = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} a_i = 0$, since the sum of the $c$'s and the sum of the $a$'s are each the sum of all the elements of $A$. The problem is to show that conversely given a function $\phi(i) = b_i$, $i = 1, \ldots, n$, with values $b_i$ in $A$ subject only to the condition that $\sum_{i=1}^{n-1} b_i = 0$, then there exists a permutation

$$(a_1, \ldots, a_n)$$

$$(c_1, \ldots, c_n)$$

of the elements of $A$ such that $c_i - a_i = b_i$, $i = 1, \ldots, n$, if the $b$'s are appropriately renumbered. This problem$^1$ is solved in this paper.

2. Solution of the problem.

Theorem. Given a function $\phi(i) = b_i$, $i = 1, \ldots, n$, with $b_i$ in $A$, an additive abelian group of order $n$, subject to the condition $\sum_{i=1}^{n-1} b_i = 0$, there exists a permutation

$$(a_1, \ldots, a_n)$$

$$(c_1, \ldots, c_n)$$

of the elements of $A$ such that $c_i - a_i = b_i$, $i = 1, \ldots, n$, the $b$'s being appropriately renumbered.

Proof. If we take $a_1, a_2, \ldots, a_n$ as the elements of $A$ in an arbitrary but fixed order, the problem consists in renumbering the $b$'s so that $a_1 + b_1 = c_1, a_2 + b_2 = c_2, \ldots, a_n + b_n = c_n$ are all distinct.

It is sufficient to prove that given a permutation whose differences are $b_1, b_2, \ldots, b_{n-2}, b_{n-1}, b_n$, we can find another whose differences $b_1, b_2, \ldots, b_{n-2}, b_{n-1}, b_n$ are the same except that two of them, $b_{n-1}'$ and $b_n'$, have been replaced by two others, $b_{n-1}$ and $b_n$, with the

$^1$ For the cyclic group this shows the truth of a conjecture of Dr. George Cramer.
same sum \( b_{n-1} + b_n = b'_{n-1} + b' \). For the identical permutation has differences 0, 0, \( \cdots \), 0 and we may replace these differences two at a time to give differences \( b_1, w_2, 0, \cdots, 0, b_2, w_3, 0, \cdots, 0; \cdots; b_1, b_2, \cdots, b_{n-1}, w_n \) where \( w_2 = -b_1, w_3 = -b_1 - b_2, \cdots, w_n = -b_1 - b_2 - \cdots - b_{n-1} = b_n \).

Thus we suppose given an incomplete permutation

\[
\begin{pmatrix}
  a_1, & \cdots, & a_{n-2}, & a_n \\
  c_1, & \cdots, & c_{n-2}, & c_n
\end{pmatrix}
\]

with differences \( b_1, b_2, \cdots, b_{n-2} \) which we represent by a table:

\[
\begin{array}{cccc}
  a_1 & a_2 & \cdots & a_{n-2} & a_n \\
  c_1 & c_2 & \cdots & c_{n-2} & c_n \\
  b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} & b_n \\
  c_1 & c_2 & \cdots & c_{n-2} & u_{n-1} & u_0.
\end{array}
\]

In this table \( a_i + b_i = c_i, \; i = 1, \cdots, n-2 \), and we have left over two \( a \)'s, two \( b \)'s, and the two elements \( u_0 \) and \( u_{-1} \) which together with \( c_1, c_2, \cdots, c_{n-2} \) make up all the elements of \( A \). Here we have

\[
\sum_{i=1}^{n-2} a_i + a_{n-1} + a_n + \sum_{i=1}^{n-2} b_i + b_{n-1} + b_n = \sum_{i=1}^{n-2} c_i + u_{-1} + u_0
\]

since each of \( \sum_{i=1}^{n-2} a_i \) and \( \sum_{i=1}^{n-2} c_i + u_{-1} + u_0 \) is the sum of all the elements of \( A \) and by hypothesis \( \sum_{i=1}^{n-1} b_i = 0 \). But since \( a_i + b_i = c_i, \; i = 1, \cdots, n-2 \), we shall have from (2.2)

\[
(2.3) \quad a_{n-1} + a_n + b_{n-1} + b_n = u_{-1} + u_0.
\]

In (2.3) if one \( a \) plus one \( b \) is one of the \( u \)'s, then the other \( a \) plus the other \( b \) is the remaining \( u \) and we can complete (2.1) to a full permutation with differences \( b_1, \cdots, b_n \) as was to be done. If not, then the equation \( x + b_{n-1} = u_0 \) has as its solution \( x = a_{r_1}, \; 1 \leq r_1 \leq n-2 \). Now in (2.1) let us replace \( b_{r_1} \) and \( c_{r_1} \) by \( b_{n-1} \) and \( u_{-1} \) leading to the following table:

\[
\begin{array}{cccc}
  a_1 & \cdots & a_{r_1} & \cdots & a_{n-2} & a_n \\
  c_1 & \cdots & u_{-1} & \cdots & c_{n-2} & u_0 \; c_{r_1} \\
  b_1 & \cdots & b_{n-1} & \cdots & b_{n-2} & b_{r_1} & b_n \\
\end{array}
\]

and as from (2.1) we have

\[
(2.5) \quad a_{n-1} + a_n + b_{r_1} + b_n = u_0 + c_{r_1}.
\]

In (2.5) if one \( a \) plus one \( b \) is \( u_0 \) or \( c_{r_1} \), the same holds for the other \( a, b, \) and \( c_{r_1} \) or \( u_0 \), and we have found a solution to the problem. If
not, the equation $x + b_{r_1} = c_{r_1}$ has a solution $x = a_{r_2}$ with $1 \leq r_2 \leq n - 2$. Let us then replace $b_{r_2}$ and $c_{r_2}$ by $b_{r_1}$ and $c_{r_1}$ in (2.4) leading to another incomplete permutation. If we continue this process for $i$ steps, we have (if $a_{r_1}, \ldots, a_{r_i}$ are all different)

$$a_1 \cdots a_{r_1} \cdots a_{r_i} \cdots a_{n-2} a_{n-1} a_n$$

(2.6)  $$b_1 \cdots b_{r_{i-1}} b_{r_i} \cdots b_{n-2} b_{r_{i-1}} b_{r_i} b_n$$

$$c_1 \cdots c_{r_{i-1}} c_{r_i} \cdots c_{n-2} c_{r_{i-1}} c_{r_i}.$$

At the $i$th stage we solve the equation $x + b_{r_i} = c_{r_{i-1}}$. If this $x$ is $a_{n-1}$ or $a_n$, the relation

$$a_{n-1} + a_n + b_{r_i} + b_n = c_{r_{i-1}} + c_{r_i}$$

(2.7)  

leads to a solution of the problem. If not, $x = a_{r_{i+1}}$ with $1 \leq r_{i+1} \leq n - 2$ and we proceed to the $(i + 1)$th stage by replacing $b_{r_{i+1}}$ and $c_{r_{i+1}}$ by $b_{r_i}$ and $c_{r_{i-1}}$. Hence either (1) we reach a solution of the problem or (2) the process continues indefinitely. We shall show that the second alternative cannot arise. In the second alternative since $a_{r_1}, a_{r_2}, \ldots$ are drawn from the finite set $a_1, \ldots, a_{n-2}$, there will be indices $i$ and $j \geq i$ such that $a_{r_1}, \ldots, a_{r_i}, \ldots, a_{r_j}$ are all distinct, but $a_{r_{j+1}} = a_{r_i}$. Then at the $j$th stage we have

$$a_1 \cdots a_{r_i} \cdots a_{r_j} \cdots a_{n-2} a_{n-1} a_n$$

(2.8)  $$b_1 \cdots b_{r_{j-1}} b_{r_{j-1}} \cdots b_{n-2} b_{r_{j-1}} b_{r_j} b_n$$

$$c_1 \cdots c_{r_{j-1}} c_{r_{j-1}} \cdots c_{n-2} c_{r_{j-1}} c_{r_j}$$

and the solution of $x + b_{r_j} = c_{r_{j-1}}$ is $x = a_{r_i}$. At the $(j + 1)$th stage the $b$'s and $c$'s left over are

$$b_{r_{i-1}} b_n$$

$$c_{r_j} c_{r_{j-1}}$$

(2.9)  

whence

$$a_{n-1} + a_n + b_{r_{i-1}} + b_n = c_{r_j} + c_{r_{i-1}}.$$  

(2.10)  

But at the $(i - 1)$th stage we had (from (2.7) or (2.3) if $i = 1$)

$$a_{n-1} + a_n + b_{r_{i-1}} + b_n = c_{r_{i-2}} + c_{r_{i-1}}.$$  

(2.11)  

Comparing (2.10) and (2.11) we find that

$$c_{r_j} = c_{r_{i-1}}.$$  

(2.12)  

But this is a contradiction since $j > i - 1$ and $c_{r_j}$ and $c_{r_{i-1}}$ are distinct elements in (2.8). Thus the second alternative does not arise and we
find a solution to the problem in not more than \( n - 2 \) steps.

3. **Application to Latin squares.** Consider a Latin square which is the Cayley table for an abelian group of order \( n \)

\[
\begin{array}{cccc}
  a_{11}, & a_{12}, & \cdots, & a_{1n} \\
  a_{21}, & a_{22}, & \cdots, & a_{2n} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{n1}, & a_{n2}, & \cdots, & a_{nn}.
\end{array}
\]  

(3.1)

Here if \( a_1 = 0, a_2, \cdots, a_n \) are the elements of \( A \), then in the table above \( a_{ij} = a_i + a_j \). If

\[
\begin{pmatrix}
  a_1, & \cdots, & a_n \\
  c_1, & \cdots, & c_n
\end{pmatrix}
\]

is a permutation of the elements of \( A \), then \( c_r \) is below \( a_r \) in the \( k \)th row if \( c_r - a_r = b_r = a_b \). We say that \( c_1, c_2, \cdots, c_r, \cdots, c_n \) agrees with the \( k \)th row in position \( r \). Thus the theorem asserts that there exists a permutation agreeing with the \( i \)th row \( k_i \) times if and only if

(3.2.1) \[ k_1 + k_2 + \cdots + k_n = n, \]

and

(3.2.2) \[ k_1 a_1 + k_2 a_2 + \cdots + k_n a_n = 0, \]

where (3.2.1) is a count of the \( k \)'s and (3.2.2) is an equation in \( A \). The sum of all the elements of an abelian group \( A \) is known to be 0 unless \( A \) contains a unique element of order 2, in which case the sum is this unique element. In the special case in which \( k_1 = k_2 = \cdots = k_n = 1 \) we say that \( c_1, \cdots, c_n \) is a transversal of the Latin square. Here (3.2.2) does not hold if \( A \) contains a unique element of order 2 and there is no transversal. But if \( A \) does not contain a unique element of order 2, then (3.2.2) does hold and there is a transversal of the Latin square. This special case of the theorem above was proved by Lowell Paige in his doctoral dissertation at the University of Wisconsin.

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