1. Introduction. Suppose we are given a finite abelian group $A$ of order $n$, the group operation being addition. If

$$\begin{pmatrix} a_1, a_2, \cdots, a_n \\ c_1, c_2, \cdots, c_n \end{pmatrix}$$

is a permutation of the elements of $A$, then the differences $c_i - a_i = b_1, \cdots, c_n - a_n = b_n$ are $n$ elements of $A$, not in general distinct, such that $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n} c_i - \sum_{i=1}^{n} a_i = 0$, since the sum of the $c$'s and the sum of the $a$'s are each the sum of all the elements of $A$. The problem is to show that conversely given a function $\phi(i) = b_i$, $i = 1, \cdots, n$, with values $b_i$ in $A$ subject only to the condition that

$$\sum_{i=1}^{n} b_i = 0,$$

then there exists a permutation

$$\begin{pmatrix} a_1, \cdots, a_n \\ c_1, \cdots, c_n \end{pmatrix}$$

of the elements of $A$ such that $c_i - a_i = b_i$, $i = 1, \cdots, n$, if the $b$'s are appropriately renumbered. This problem$^1$ is solved in this paper.

2. Solution of the problem.

Theorem. Given a function $\phi(i) = b_i$, $i = 1, \cdots, n$, with $b_i$ in $A$, an additive abelian group of order $n$, subject to the condition $\sum_{i=1}^{n} b_i = 0$, there exists a permutation

$$\begin{pmatrix} a_1, \cdots, a_n \\ c_1, \cdots, c_n \end{pmatrix}$$

of the elements of $A$ such that $c_i - a_i = b_i$, $i = 1, \cdots, n$, the $b$'s being appropriately renumbered.

Proof. If we take $a_1, a_2, \cdots, a_n$ as the elements of $A$ in an arbitrary but fixed order, the problem consists in renumbering the $b$'s so that

$$a_1 + b_1 = c_1, a_2 + b_2 = c_2, \cdots, a_n + b_n = c_n$$

are all distinct.

It is sufficient to prove that given a permutation whose differences are $b_1, b_2, \cdots, b_{n-2}, b_{n-1}'$, we can find another whose differences $b_1, b_2, \cdots, b_{n-2}, b_{n-1}, b_n$ are the same except that two of them, $b_{n-1}'$ and $b_n'$, have been replaced by two others, $b_{n-1}$ and $b_n$, with the

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$^1$ For the cyclic group this shows the truth of a conjecture of Dr. George Cramer.
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same sum $b_{n-1} + b_n = b'_{n-1} + b'$. For the identical permutation has differences 0, 0, \cdots, 0 and we may replace these differences two at a time to give differences $b_1$, $w_2$, 0, \cdots, 0; $b_1$, $b_2$, $w_3$, 0, \cdots, 0; \cdots; b_1$, $b_2$, \cdots, $b_{n-1}$, $w_n$ where $w_2 = -b_1$, $w_3 = -b_1 - b_2$, \cdots, $w_n = -b_1 - b_2 - \cdots - b_{n-1} = b_n$.

Thus we suppose given an incomplete permutation
\[
\begin{pmatrix} a_1, \cdots, a_{n-2}, \cdots \end{pmatrix} \\
\begin{pmatrix} c_1, \cdots, c_{n-2}, \cdots \end{pmatrix}
\]
with differences $b_1$, $b_2$, \cdots, $b_{n-2}$ which we represent by a table:
\[
\begin{array}{cccc}
\begin{array}{c}
\beta_1 \\
\beta_i \\
\beta_{n-2} \\
\end{array} & \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_{n-2} \\
\end{array} & \begin{array}{c}
a_2 \\
a_{n-2} \\
a_n \\
\end{array} & \\
\begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\end{array} & \begin{array}{c}
\beta_2 \\
\beta_{n-2} \\
\beta_n \\
\end{array} & \begin{array}{c}
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
b_{n-1} \\
b_n \\
\gamma_0 \\
\end{array} \\
\begin{array}{c}
c_1 \\
c_2 \\
c_{n-2} \\
\end{array} & \begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\end{array} & \begin{array}{c}
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\end{array} & \begin{array}{c}
u_0 \\
u_{n-1} \\
u_n \\
\end{array}
\end{array}
\] (2.1)

In this table $a_i + \beta_i = c_i$, $i = 1$, \cdots, $n-2$, and we have left over two $a$'s, two $b$'s, and the two elements $u_0$ and $u_{-1}$ which together with $c_1$, $c_2$, \cdots, $c_{n-2}$ make up all the elements of $A$. Here we have
\[
\sum_{i=1}^{n-2} a_i + a_{n-1} + a_n + \sum_{i=1}^{n-2} \beta_i + b_{n-1} + b_n = \sum_{i=1}^{n-2} c_i + u_{-1} + u_0
\] (2.2)

since each of $\sum_{i=1}^{n-2} a_i$ and $\sum_{i=1}^{n-2} \beta_i + u_{-1} + u_0$ is the sum of all the elements of $A$ and by hypothesis $\sum_{i=1}^{n} b_i = 0$. But since $a_i + \beta_i = c_i$, $i = 1$, \cdots, $n-2$, we shall have from (2.2)
\[
a_{n-1} + a_n + b_{n-1} + b_n = u_{-1} + u_0.
\] (2.3)

In (2.3) if one $a$ plus one $b$ is one of the $u$'s, then the other $a$ plus the other $b$ is the remaining $u$ and we can complete (2.1) to a full permutation with differences $b_1$, \cdots, $b_n$ as was to be done. If not, then the equation $\sum_{i=1}^{n-2} a_i + a_{n-1} + a_n + \sum_{i=1}^{n-2} \beta_i + b_{n-1} + b_n = \sum_{i=1}^{n-2} c_i + u_{-1} + u_0$ has as its solution $x = a_r$, $1 \leq r \leq n-2$. Now in (2.1) let us replace $b_r$ and $c_r$ by $b_{n-1}$ and $u_{-1}$ leading to the following table:
\[
\begin{array}{cccc}
\begin{array}{c}
a_1 \\
a_2 \\
a_{n-2} \\
a_{n-1} \\
a_n \\
\end{array} & \begin{array}{c}
a_1 \\
a_2 \\
a_{n-2} \\
a_{n-1} \\
a_n \\
\end{array} & \begin{array}{c}
a_1 \\
a_2 \\
a_{n-2} \\
a_{n-1} \\
a_n \\
\end{array} & \begin{array}{c}
a_1 \\
a_2 \\
a_{n-2} \\
a_{n-1} \\
a_n \\
\end{array} \\
\begin{array}{c}
\beta_1 \\
\beta_2 \\
\beta_{n-2} \\
\beta_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
\beta_1 \\
\beta_2 \\
\beta_{n-2} \\
\beta_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
\beta_1 \\
\beta_2 \\
\beta_{n-2} \\
\beta_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
\beta_1 \\
\beta_2 \\
\beta_{n-2} \\
\beta_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} \\
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_{n-2} \\
\alpha_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_{n-2} \\
\alpha_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_{n-2} \\
\alpha_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_{n-2} \\
\alpha_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} \\
\begin{array}{c}
c_1 \\
c_2 \\
c_{n-2} \\
c_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
c_1 \\
c_2 \\
c_{n-2} \\
c_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
c_1 \\
c_2 \\
c_{n-2} \\
c_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array} & \begin{array}{c}
c_1 \\
c_2 \\
c_{n-2} \\
c_n \\
\gamma_1 \\
\gamma_2 \\
\gamma_{n-2} \\
\gamma_n \\
\end{array}
\end{array}
\] (2.4)

and as from (2.1) we have
\[
a_{n-1} + a_n + b_{r_1} + b_n = u_0 + c_{r_1}.
\] (2.5)

In (2.5) if one $a$ plus one $b$ is $u_0$ or $c_{r_1}$, the same holds for the other $a$, $b$, and $c_{r_1}$ or $u_0$ and we have found a solution to the problem. If
not, the equation \( x + b_{r_1} = u_0 \) has a solution \( x = a_{r_2} \) with \( 1 \leq r_2 \leq n - 2 \). Let us then replace \( b_{r_2} \) and \( c_{r_2} \) by \( b_{r_1} \) and \( u_0 \) in (2.4) leading to another incomplete permutation. If we continue this process for \( i \) steps, we have (if \( a_{r_1}, \ldots, a_{r_i} \) are all different)

\[
\begin{align*}
& a_1 \cdots a_{r_1} a_{r_2} \cdots a_{r_i} \cdots a_{n-2} a_{n-1} a_n \\
& b_1 \cdots b_{n-1} b_{r_1} b_{r_2} \cdots b_{r_i-1} b_{n-2} b_{r_i} b_n \\
& c_1 \cdots c_{n-1} c_{r_1} \cdots c_{r_i-2} \cdots c_{n-2} c_{r_i} c_{r_i}.
\end{align*}
\]

At the \( i \)th stage we solve the equation \( x + b_{r_i} = c_{r_i-1} \). If this \( x \) is \( a_{n-1} \) or \( a_n \), the relation

\[
(2.7) \quad a_{n-1} + a_n + b_{r_i} + b_n = c_{r_i-1} + c_{r_i}
\]

leads to a solution of the problem. If not, \( x = a_{r_{i+1}} \) with \( 1 \leq r_{i+1} \leq n - 2 \) and we proceed to the \((i+1)\)th stage by replacing \( b_{r_{i+1}} \) and \( c_{r_{i+1}} \) by \( b_{r_i} \) and \( c_{r_i-1} \). Hence either (1) we reach a solution of the problem or (2) the process continues indefinitely. We shall show that the second alternative cannot arise. In the second alternative since \( a_{r_1}, a_{r_2}, \ldots \) are drawn from the finite set \( a_1, \ldots, a_{n-2} \), there will be indices \( i \) and \( j \geq i \) such that \( a_{r_1}, \ldots, a_{r_i}, \ldots, a_{r_j} \) are all distinct, but \( a_{r_{j+1}} = a_{r_i} \). Then at the \( j \)th stage we have

\[
\begin{align*}
& a_1 \cdots a_{r_j} \cdots a_{r_i} \cdots a_{n-2} a_{n-1} a_n \\
& b_1 \cdots b_{r_{j-1}} \cdots b_{r_{j-2}} \cdots b_{n-2} b_{r_j} b_n \\
& c_1 \cdots c_{r_{j-2}} \cdots c_{r_{j-3}} \cdots c_{n-2} c_{r_j} c_{r_j}.
\end{align*}
\]

and the solution of \( x + b_{r_j} = c_{r_{j-1}} \) is \( x = a_{r_j} \). At the \((j+1)\)th stage the \( b \)'s and \( c \)'s left over are

\[
(2.9) \quad b_{r_{i+1}} b_n \\
\]

whence

\[
(2.10) \quad a_{n-1} + a_n + b_{r_{i+1}} + b_n = c_{r_j} + c_{r_{i-1}}.
\]

But at the \((i-1)\)th stage we had (from (2.7) or (2.3) if \( i = 1 \))

\[
(2.11) \quad a_{n-1} + a_n + b_{r_{i-1}} + b_n = c_{r_{i-2}} + c_{r_{i-1}}.
\]

Comparing (2.10) and (2.11) we find that

\[
(2.12) \quad c_{r_j} = c_{r_{i-1}}.
\]

But this is a contradiction since \( j > i - 1 \) and \( c_{r_j} \) and \( c_{r_{i-1}} \) are distinct elements in (2.8). Thus the second alternative does not arise and we
find a solution to the problem in not more than \( n - 2 \) steps.

3. **Application to Latin squares.** Consider a Latin square which is the Cayley table for an abelian group of order \( n \)

\[
\begin{array}{cccc}
a_{11}, & a_{12}, & \cdots, & a_{1n} \\
a_{21}, & a_{22}, & \cdots, & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}, & a_{n2}, & \cdots, & a_{nn}.
\end{array}
\]

(3.1)

Here if \( a_1 = 0, a_2, \cdots, a_n \) are the elements of \( A \), then in the table above \( a_{ij} = a_i + a_j \). If

\[
\left( a_1, \cdots, a_n \right) \\
\left( c_1, \cdots, c_n \right)
\]

is a permutation of the elements of \( A \), then \( c_r \) is below \( a_r \) in the \( k \)th row if \( c_r - a_r = b_r = a_b \). We say that \( c_1, c_2, \cdots, c_r, \cdots, c_n \) agrees with the \( k \)th row in position \( r \). Thus the theorem asserts that there exists a permutation agreeing with the \( i \)th row \( k_i \) times if and only if

\[
k_1 + k_2 + \cdots + k_n = n,
\]

and

\[
k_1a_1 + k_2a_2 + \cdots + k_na_n = 0,
\]

where (3.2.1) is a count of the \( k \)'s and (3.2.2) is an equation in \( A \). The sum of all the elements of an abelian group \( A \) is known to be 0 unless \( A \) contains a unique element of order 2, in which case the sum is this unique element. In the special case in which \( k_1 = k_2 = \cdots = k_n = 1 \) we say that \( c_1, \cdots, c_n \) is a transversal of the Latin square. Here (3.2.2) does not hold if \( A \) contains a unique element of order 2 and there is no transversal. But if \( A \) does not contain a unique element of order 2, then (3.2.2) does hold and there is a transversal of the Latin square. This special case of the theorem above was proved by Lowell Paige in his doctoral dissertation at the University of Wisconsin.

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