THE QUADRIC OF LIE

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1. Introduction. In this paper we describe a simple way of generating the quadric of Lie. Similar methods of generating the quadrics of Lane [7], of Wilczynski [8] and Fubini, and the quadrics $Q(l_2, m_2, l_3, m_3)$ described previously by us [3] are described. In the description of the method of generating these quadrics another description of the curves corresponding to the developables of reciprocal congruences is found.

Let the homogeneous projective coordinates $(x^1, x^2, x^3, x^4)$ of a generic point $x$ of a surface $S$ be functions of the asymptotic parameters $u, v$ so normalized [4] that they satisfy the differential equations

$$XuU = OXu,$$  
$$Xv = yxu + 0vxv + qx,  
$$0 = \log R.$$

We have called the line determined by the points $x'x''$ the $R$-harmonic line, and that determined by $x, x''$ the $R$-conjugate line.

If we define the homogeneous projective coordinates $\xi$ of the tangent plane by the expression

$$\xi = R^{-1}(x, x_u, x_v),$$

it follows that the functions $\xi$ satisfy the differential equations [6]

$$\xi_{uu} = \theta_{u} \xi_{u} - \beta_{v} \xi_{v} + \pi x,  
\xi_{vv} = -\gamma \xi_{u} + \theta_{u} \xi_{v} + \chi x,$$

wherein

$$\pi = \rho + \beta_{v} + \beta \theta_{v}, \quad \chi = q + \gamma u + \gamma \theta u.$$

It is well known [2] that a geometric characterization can be given for reciprocal lines associated with $S$ at $x$ without using the quadrics of Darboux with respect to which the lines are reciprocal. Let a line $l_2$ in the tangent plane to $S$ at $x$ be determined by the points $r, s$ whose coordinates are defined by the expressions

$$r = x_u - bx, \quad s = x_v - ax.$$

A point $r_v$ on the tangent to the locus of $r$ as $x$ generates the asymp-
totic curve \( u = \text{const.} \) has coordinates
\[ r = x_u - bs - (ab + b_s)x. \]
Similarly the point \( s_u \) with coordinates
\[ s = x_u - ar - (ab + a_u)x \]
is found.

The common intersector \( l_1 \) of the tangents to the loci of the points \( r \) and \( s \) which passes through \( x \) intersects these respective tangents in the points \( z_1, z_2 \) defined by the expressions
\[ z_1 = x_u - ax_u - bx_v + (ab - bv)x, \]
\[ z_2 = x_u - ax_u - bx_v + (ab - au)x. \]
The lines \( l_1, l_2 \) are reciprocal lines. Moreover the quadrics of Darboux may be characterised as the only quadrics through \( x \) having \( l_1, l_2 \) as reciprocal lines for all lines \( l_2 \) in the tangent plane to \( S \) at \( x \).

The harmonic conjugate \( h \) of \( x \) with respect to the points \( z_1, z_2 \) has coordinates
\[ h = x_u - ax_u - bx_v + [ab - 2^{-1}(a_u + b_v)]x. \]
We shall call \( h \) the harmonic point on \( l_1 \). The points \( z_1, z_2 \) (and \( h \)) coincide if and only if \( a_u - b_v = 0 \), that is, if and only if the congruence \( \Gamma_2 \) of lines \( l_2 \) is central \([1]\) to \( S \).

2. A family of projectivities on \( l_2 \). Let \( z \) be any point on \( l_1 \) (except \( x \)) with coordinates of the form
\[ z = x_u - ax_u - bx_v + \phi x. \]
As \( x \) generates the curve \( C \) through \( x \) of the one-parameter family of curves defined by
\[ \mu du - \lambda dv = 0, \]
the point \( z \) describes a curve. The tangent of this curve intersects the tangent plane to \( S \) at \( x \) in a point whose projection \( \tilde{y} \) from \( x \) on the line \( l_2 \) has coordinates
\[ \tilde{y} = \tilde{\lambda}r + \tilde{\mu}s \]
wherein
\[ \tilde{\lambda} = (\kappa - ab - a_u + \phi)\lambda + (\chi - a_v - b\gamma - a^2)\mu, \]
\[ \tilde{\mu} = (\sigma - b_u - a\beta - b^2)\lambda + (\kappa - ab - b_v + \phi)\mu, \]
\[ \kappa = \beta\gamma + \theta_{uv}. \]
The tangent to $C$ at $x$ intersects $l_2$ in a point $y$ with coordinates $y = \lambda r + \mu s$.

Formulas (2.3) therefore represent a family $F(\phi)$ of projectivities on $l_2$, $y$ and $\hat{y}$ being corresponding points in any particular member of the family.

In $F(\phi)$ there is one involution determined by
\begin{equation}
\phi = ab + 2^{-1}(a_u + b_v) - \kappa.
\end{equation}

The point $I$, defined by (2.1) and (2.4), will be called the involutory point on $l_i$. Its coordinates $I_i$ are defined by the expression
\begin{equation}
I_i = x_{uv} - ax_u - bx_v + [ab + 2^{-1}(a_u + b_v) - \kappa]x.
\end{equation}

A comparison of (2.5) with the coordinates of the focal points [6] on $l_i$ shows that the involutory point on $l_i$ is the harmonic conjugate of $x$ with respect to the focal points.

Let $g$ be the harmonic conjugate of $x$ with respect to the harmonic point $h$, and the involutory point $I_i$. The coordinates of $g$ are readily found to be given by the formula
\begin{equation}
g = x_{uv} - ax_u - bx_v + (ab - 2^{-1}\kappa)x.
\end{equation}

Defining the local coordinates $(x_1, x_2, x_3, x_4)$ of a point $X$ by the expression
\begin{equation}
X = x_1x + x_2x_u + x_3x_v + x_4x_{uv},
\end{equation}
we find that the locus of the point $g$ as $h$ varies in the tangent plane has the equation
\begin{equation}
x_2x_3 - x_1x_4 - 2^{-1}(\theta u) x_4^2 = 0.
\end{equation}

That is, the locus of $g$ is the quadric of Lie of $S$ at $x$. The lines $(g_r)$ and $(g_s)$ are generators of this quadric, and hence the quadric of Lie is enveloped by the plane of $g$ and $l_2$. We shall call the point $g$ the generating point of $l_i$.

Returning to the family $F(\phi)$ of projectivities defined by (2.3) we observe that the double points of these projectivities are identical for all members of the family. These double points are given by the quadratic equation
\begin{equation}
(x - b_u - a\beta - b^2)\lambda^2 - (b_v - a_u)\lambda\mu - (x - a_v - b\gamma - a^2)\mu^2 = 0.
\end{equation}

That is, the double points of the projectivities of $F(\phi)$ are the intersections of the tangents to the $\Gamma_1$-curves of the congruence $\Gamma_1$ of lines $l_i$ with $l_2$. Hence another interpretation may be placed on the curves
corresponding to the developables of a congruence \( \Gamma_1 \) of lines protruding from a surface.

The above constructions may be dualized. Any plane \( \xi \) through \( l_2 \) has coordinates of the form

\[
\xi = \xi_u - a\xi - b\xi_v + \phi\xi.
\]

As \( x \) moves along the curve \( C \) defined by (2.2), the plane \( \xi \) envelops a developable surface whose generator in \( \xi \) determines with \( x \) a plane whose line of intersection with the tangent plane \( \xi \) and \( l_1 \) determine a plane whose coordinates are \( \xi' + \mu \alpha \) wherein \( \alpha = \xi_u - b\xi \), \( \beta = \xi_v - a\xi \) and

\[
\begin{align*}
\lambda &= (\kappa + ab - a\alpha + \phi)\lambda + (q - a\alpha + b\gamma - a^2)\mu, \\
\mu &= (p - b\alpha + a\beta - b^2)\lambda + (\kappa + ab - b\gamma + \phi)\mu.
\end{align*}
\]

The double planes of the family (2.8) of projectivities correspond to the developables of the congruence \( \Gamma_2 \) of lines \( l_2 \).

Among the projectivities (2.8) is one involution determined by the plane through \( l_2 \) with coordinates

\[
\xi' + \mu\alpha = quv - a\xi - b\xi_v + [ab + 2^{-1}(a\alpha + b\gamma) - k]x.
\]

We shall call this plane the involutory plane through \( l_2 \). The involutory plane intersects \( l_1 \) in a point \( I_2 \) whose coordinates have the form

\[
I_2 = x' + a\xi - b\xi_v + [ab - 2^{-1}(a\alpha + b\gamma) - k]x.
\]

The harmonic conjugate of \( x \) with respect to the involutory point \( I_1 \) and the point \( I_2 \) is the generating point \( g \).

Similarly an enveloping plane (the dual of the generating point \( g \)) may be described. Its coordinates are

\[
\xi' + \mu\alpha = quv - a\xi - b\xi_v + (ab - 2^{-1}\kappa)\xi.
\]

The envelope of this enveloping plane is of course the quadric of Lie.

3. Quadrics associated with the surface. Some of the notions developed in the paper may be used to describe methods of generating other quadrics associated with a surface at a point. Before describing these methods, we first give an extension of the \( R_\lambda \)-associate of a line \( l_4 \) and the \( R_\lambda \)-derived line and curves developed by Bell.

Let there be given on \( S \) two one-parameter families of curves, not necessarily distinct. The differential equations of such a set of curves may be written in the form

\[
(du - \lambda dv)(du - \mu dv) = 0.
\]
Denote the curves of these families through \( x \) by \( C_{\lambda}, C_{\mu} \) respectively. As \( x \) generates the curve \( C_{\lambda} \) the point \( r \) defined by (1.2) describes a curve. Similarly the point \( s \) describes a curve as \( x \) moves on \( C_{\mu} \).

The common intersector \( l_{1,\lambda} \) through \( x \) of the tangents to these loci of \( r \) and \( s \) joins \( x \) to the point \( z \) whose coordinates are

\[
(3.2) \quad z = x_{uv} - (a - \gamma \lambda)x_u - (b - \beta \mu)x_v.
\]

The reciprocal of \( l_{1,\lambda} \) is the line \( l_{2,\lambda} \) joining the points \( r_{\lambda}, s_{\lambda} \) defined by the formulas

\[
(3.3) \quad r_{\lambda} = r + \beta \mu x, \quad s_{\lambda} = s + \gamma \lambda x.
\]

We shall call the line \( l_{2,\lambda} \) the \( R_{\lambda}-\text{associate} \) of \( l_2 \). If in its definition the curves \( C_{\lambda} \) and \( C_{\mu} \) are identical, the \( R_{\lambda}-\text{associate} \) is the \( R_{\lambda}-\text{associate} \) of \( l_2 \) as defined [1] by Bell.

Bell has called [1] the line joining \( x \) to the intersection of the \( R_{\lambda}-\text{associate} \) of \( l_2 \) with \( l_2 \) the \( R_{\lambda}-\text{derived} \) line. Similarly we call the line determined by \( x \) and the intersection of the \( R_{\lambda}-\text{associate} \) of \( l_2 \) with \( l_2 \) the \( R_{\lambda}-\text{derived} \) line. This line has the direction \( dv/du \) defined by

\[
(3.4) \quad \gamma dv - \beta du = 0.
\]

We may readily verify that the tangent to \( C \) at \( x \) and the \( R_{\lambda}-\text{derived} \) line are conjugate directions if and only if \( \mu = -\gamma \lambda^2/\beta \), that is, if and only if the tangent to \( C_{\lambda} \) is the \( R_{\lambda}-\text{derived} \) line [1] of Bell.

From (2.6) and (3.3) it is readily seen that the generating point \( g_{\lambda} \) on \( l_{1,\lambda} \) has coordinates given by the formula

\[
g_{\lambda} = x_{uv} - (a - \gamma \lambda)x_u - (b - \beta \mu)x_v + [(a - \gamma \lambda)(b - \beta \mu) - 2^{-1}k]x.
\]

The plane determined by \( l_2 \) and \( g_{\lambda} \) intersects \( l_1 \) (the reciprocal of \( l_2 \)) in the point \( g^*_{\lambda} \) defined by

\[
g^*_{\lambda} = x_{uv} - ax_u - bx_v + [ab + \beta \gamma \mu - 2^{-1}k]x.
\]

The locus of \( g^*_{\lambda} \) as \( l_2 \) varies in the tangent plane to \( S \) at \( x \) is the quadric \( Q_{\lambda \mu} \) whose equation is

\[
(3.5) \quad x_2x_3 - x_1x_4 - 2^{-1}[(1 - h)\beta \gamma + \theta_{uv}]x_4^2 = 0, \quad h = 2\lambda \mu.
\]

One of the cross ratios of the asymptotic tangents and the tangents to \( C_{\lambda}, C_{\mu} \) is \( \lambda \mu \). If we impose the condition that this cross ratio be constant, the quadrics \( Q_{\lambda \mu} \) becomes the quadrics [7] of Lane. If either one or both of \( C_{\lambda}, C_{\mu} \) is an asymptotic curve, the quadric \( Q_{\lambda \mu} \) is the quadric of Lie. If the tangents to \( C_{\lambda} \) and \( v=\text{const} \) separate the tangents to \( C_{\mu} \) and \( u=\text{const} \) harmonically, \( Q_{\lambda \mu} \) is the quadric of...
Wilczyński. If \( C_x, C_u \) are curves of a conjugate net, the quadric \( Q_{s,u} \) could be called the *conjugate quadric of \( S \) at \( x \). Its equation is

\[
x_3x_2 - x_1x_4 - 2^{-1}[3\beta y + 0\upsilon]x_4 = 0.
\]

We have previously [3] characterized a line called the \( (l_3, m_3, l_3, m_3) \)-associate of \( l_3 \) in terms of constant cross ratios and involving the \( R_\lambda \)-associate and \( R_\lambda \)-derived line of Bell. If we proceed in a similar manner, a generalized \( (l_3, m_3, l_3, m_3) \)-associate of \( l_3 \) can be characterized using constant cross ratios and replacing the \( R_\lambda \)-associate and \( R_\lambda \)-derived lines by the \( R_{\lambda, u} \)-associate and \( R_{\lambda, u} \)-derived lines respectively. Such a generalized associate of \( l_3 \) is determined by the formulas

\[
(3.6) \quad r_{\lambda, u} = x_u - (b - k_2)x, \quad s_{\lambda, u} = x_v - (a - k_3)x
\]

wherein

\[
k_2 = l_3\beta\mu + m_3\gamma\lambda/\mu, \quad k_3 = l_3\beta\mu/\lambda + m_3\gamma\lambda, \quad k, m_2, l_3, m_3 \text{ being constants.}
\]

Hence if we use the special \((0, m_3, l_3, 0)\)-associate (or \((l_3, 0, 0, m_3)\)-associate) in place of the \( R_{\lambda, u} \)-associate in the above characterization of the quadrics \( R_{\lambda, u} \), the family of quadrics of Lane with \( h = l_3m_3 \) (or \( h = l_3m_3 \)) is obtained.

Finally it is easy to verify that the plane determined by the \( R \)-harmonic line and the generating point on the reciprocal of the above generalized \((l_3m_3, l_3m_3)\)-associate of \( l_3 \) intersects the reciprocal of \( l_3 \) in the point \( g_{\lambda, u} \) whose coordinates are given by the expression

\[
g_{\lambda, u} = x_{uv} - a x_u - b x_v + [(a - k_3)(b - k_2) - 2^{-1}k]x.
\]

The locus of this point is the quadric whose equation is

\[
x_3x_2 + x_4[-x_1 + k_2x_2 + k_3x_3 + (k_2k_3 - 2^{-1}k)x_1] = 0.
\]

This quadric is a member of a family which might be called the *generalized \((l_3, m_3, l_3, m_3)\) quadrics*. In particular if \( l_2 = m_2 = l_3 = m_3 = 0 \), the quadric is the quadric of Lie.

**References**

We extend a theorem of Rédei and Szép. Our proof is quite straightforward, and employs a method of considerably more general applicability.

The lower central series of a group $G$ is formed by taking $G_1 = G$, and successively defining $G_{n+1}$ to be the commutator $(G_n, G)$. $G$ is nilpotent if some $G_{N+1} = 1$. If $A$ and $B$ are subgroups of $G$, $A \vee B$ is the subgroup generated by the elements of $A$ and of $B$ together, and $A^m$ the subgroup generated by the $m$th powers of elements of $A$.

**Theorem.** Let $A$ and $K$ be subgroups of a nilpotent group $G$, and let $A^m = 1$ for some integer $m$. Then, for any $n \geq 1$,

$$(A \vee K)_n = (A^m \vee K)_n \implies (A \vee K)_n = K_n.$$

We may clearly suppose that $G = A \vee K$. The elements of $G_r$ can be written as products of commutators of order $r$:

$$(x_1, \ldots, x_r) = ((\cdots ((x_1, x_2), x_3) \cdots , x_{r-1}), x_r).$$

Let $C_r$ be the subgroup generated by those commutators for which

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1. L. Rédei and J. Szép, Monatshfte für Mathematik vol. 55, p. 200. The present proof avoids "counting arguments" and the attendant finiteness conditions; for $n = 1$ the present argument reduces substantially to that of Rédei and Szép. We remark that the hypothesis $A^m = 1$ admits various modifications.