A CLASS OF MULTIVALENT FUNCTIONS
WITH ASSIGNED ZEROS

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1. Introduction. Recently A. W. Goodman [1; 2] has studied the following two classes of multivalent functions:

(i) \( p \)-valently starlike functions denoted by \( S(p) \): A function \( f(z) \) is said to be \( p \)-valently starlike with respect to the origin for \( |z| < 1 \) if (a) \( f(z) \) is regular and \( p \)-valent for \( |z| < 1 \) and (b) if there exists a \( p \) such that, for each \( r \) in \( \rho < r < 1 \), the radius vector joining the origin to \( f(re^{i\theta}) \) turns continuously in the counterclockwise direction and makes \( p \) complete revolutions as \( \theta \) varies from 0 to \( 2\pi \).

(ii) Typically-real functions of order \( p \) denoted by \( T(p) \). A function

\[
f(z) = \sum_{n=0}^{\infty} b_n z^n
\]

is said to be typically-real of order \( p \) if in (1.1) the coefficients \( b_n \) are all real and if \( f(z) \) is regular in \( |z| \leq 1 \) and \( \Im(f(e^{i\theta})) \) changes sign \( 2p \) times as \( \theta \) traverses the boundary of the unit circle.

Concerning the above classes of functions he obtained the following results:

Let

\[
f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n
\]

be a function of the set \( S(p) \) or \( T(p) \). Suppose that in addition to the \( q \)th order zero at \( z=0 \), the function \( f(z) \) has exactly \( p-q \) zeros, \( \beta_1, \beta_2, \ldots, \beta_{p-q} \), such that \( 0 < |\beta_j| < 1, j = 1, 2, \ldots, p-q \). Then

\[
|a_n| \leq A_n, \quad n = q + 1, q + 2, \ldots
\]

where \( A_n \) is defined by

\[
F(z) = \frac{z^q}{(1-z)^{2p}} \prod_{j=1}^{p-q} \left( 1 + \frac{z}{|\beta_j|} \right) (1 + z |\beta_j|)
\]

\[
= z^q + \sum_{n=q+1}^{\infty} A_n z^n.
\]

The inequality (1.3) is sharp.

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For functions of the set $T(p)$ he has obtained a more general result [2; 3]. However even that result cannot include the above result for $S(p)$ since in $S(p)$ the coefficients can be complex.

Now in the present paper we shall introduce a wider class of functions $D(p)$ which includes $S(p)$, $T(p)$ and others in the case where $f(z)$ has $p$ zeros, proving that the inequality (1.3) is also valid for the functions of this class.

2. Preliminary considerations.

**Lemma 1.** Let

\[ w = f(z) = \sum_{n=0}^{\infty} a_n z^n \]

be regular for $|z| \leq 1$ and have $p$ ($\geq 0$) zeros in $|z| \leq 1$. Then there exists a point $\zeta$ (|\zeta| = 1) for which the following equality holds

\[ \arg f(-\zeta) = \arg f(\zeta) + p\pi. \]  

**Proof.** Without loss of generality, let $\arg f(-1) - \arg f(+1) < p\pi$. If a point $\zeta$ moves from $+1$ to $-1$, $\arg f(-\zeta) - \arg f(\zeta)$ varies continuously from $\arg f(-1) - \arg f(+1) < p\pi$ to $2p\pi - (\arg f(-1) - \arg f(+1)) > p\pi$, since $f(z)$ has $p$ zeros. Hence at a point $\zeta$ the equality (2.2) holds.

The special cases of Lemma 1 and the following Definition 1 we owe to N. G. DeBruijn [4] and S. Ozaki [5].

**Definition 1.** Let us say the diametral line of $f(z)$ for the straight line $[f(\zeta)f(-\zeta)]$ when $\zeta$ satisfies Lemma 1.

Accordingly we have the following:

**Lemma 1'.** Let (2.1) be a function regular for $|z| \leq 1$. Then there exists at least one diametral line of $f(z)$ in the $w$-plane.

**Definition 2.** Let $f(z)$ be regular for $|z| \leq 1$ and let $C$ be the image curve of $|z| = 1$. If $C$ is cut by a straight line passing through the origin in $2p$, and not more than $2p$ points, then $f(z)$ is said to be starlike of order $p$ in the direction of the straight line. Especially when the direction of starlikeness of order $p$ is that of the diametral line of $f(z)$, $f(z)$ is said to belong to the class $D(p)$.

The idea of being starlike in one direction was introduced by M. S. Robertson [6] and also extended to general $p$ by him [7; 8]. And $D(1)$ was studied in [4; 5].

**Lemma 2.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a member of the class $D(p)$. Further let $f(z)$ have $s$ zeros $\beta_1, \beta_2, \cdots, \beta_s$ such that $0 < |\beta_j| < 1$, $j = 1, 2, \cdots, s$. 

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Then the function $F(z)$ defined by
\[ F(z) = f(z)g(z), \quad g(z) = z^p \prod_{i=1}^{q} (z - \beta_i)(1 - \bar{\beta}_i z) \]
is also a member of the class $D(p)$. 

**Proof.** Regularity of $F(z)$ in $|z| \leq 1$ is evident. Now we easily see that
\[ g(e^{i\theta}) = \prod_{i=1}^{q} |e^{i\theta} - \beta_i|^2. \]
Hence \[ \arg F(e^{i\theta}) = \arg f(e^{i\theta}) \] for every $\theta$. Consequently if $f(z) \in D(p)$, then $F(z) \in D(p)$. 

3. **The main theorem.**

**Theorem 1.** Let
\( f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n \)
be a function of the set $D(p)$. Suppose that in addition to the $q$th order zero at $z = 0$, the function $f(z)$ has exactly $p - q$ zeros, $\beta_1, \beta_2, \ldots, \beta_{p-q}$, such that $0 < |\beta_i| < 1$, $j = 1, 2, \ldots, p-q$. Then
\begin{align*}
(3.2) \quad |a_n| &\leq B_n, \quad n = q + 1, q + 2, \ldots, \\
(3.3) \quad |f(re^{i\theta})| &\leq F(r) \quad \text{for } r < 1,
\end{align*}
where $B_n$ and $F(r)$ are defined by
\begin{align*}
F(z) &= \frac{z^q}{(1 - z)^{p-q}} \prod_{i=1}^{p-q} \left(1 + \frac{z}{|\beta_i|}(1 + z|\beta_i|)\right) \\
&= z^q + \sum_{n=q+1}^{\infty} B_n z^n.
\end{align*}

**Proof.** Let us put
\( E(z) = f(z) \cdot z^{p-q} / \prod_{i=1}^{q} (z - \beta_i)(1 - \bar{\beta}_i z). \)
Then by Lemma 2, $E(z) \in D(p)$ since $f(z) \in D(p)$, and
\begin{align*}
(-1)^{p-q} \prod_{i=1}^{q} \beta_i E(z) &= z^p + \alpha_{p+1} z^{p+1} + \cdots \\
&= \psi(z) \in D(p).
\end{align*}
We wish now to show that
\[ \psi(z) \ll z^p/(1 - z)^{2p}. \]

For the purpose it will be sufficient to assume that the diametral line in whose direction \( \psi(z) \) is starlike of order \( p \) is \( \psi(1) \equiv \psi(-1) \), since in the other cases we may consider \( \psi(\zeta z) = g(z) \) for which \( g(1) \equiv g(-1) \) is the diametral line.

Let \( \psi(1) = \omega = |\omega|e^{-ia} \); then by our hypothesis
\[ \Im e^{ia}\psi(e^{i\theta}) > 0 \quad \text{for} \quad \theta_{2s-1} < \theta < \theta_{2s}, \]
\[ \Im e^{ia}\psi(e^{i\theta}) < 0 \quad \text{for} \quad \theta_{2p} < \theta < \theta_{2p+1}, \]
\[ s = 1, 2, \ldots, p, \quad \theta_{2p+1} = \theta_1 + 2\pi, \quad \theta_1 = 0, \quad \theta_j = \pi, \quad 1 < j \leq 2p. \]

Let
\[ \phi(z) = (-1)^{p-1}\exp\left(-\frac{i}{2} \sum_{s=1}^{2p} \theta_s \right) \prod_{s=1}^{2p} \frac{e^{i\theta_s} - z}{z^p}, \]
then
\[ \phi(e^{i\theta}) = -2^{2p} \prod_{s=1}^{2p} \sin \frac{\theta_s - \theta}{2}. \]
Hence we obtain
\[ \phi(e^{i\theta}) > 0 \quad \text{for} \quad \theta_{2s-1} < \theta < \theta_{2s}, \]
\[ \phi(e^{i\theta}) < 0 \quad \text{for} \quad \theta_{2s} < \theta < \theta_{2s+1}, \]
\[ s = 1, 2, \ldots, p. \]

Let
\[ G(z) = -ie^{ia}\psi(z)\phi(z) = e^{i\theta} + \sum_{n=1}^{\infty} \gamma_n z^n, \]
then \( G(z) \) is regular for \( |z| \leq 1 \) and
\[ \Re G(e^{i\theta}) \geq 0. \]
Accordingly by the principle of minimum for regular harmonic functions
\[ \Re G(z) > 0 \quad \text{for} \quad |z| < 1. \]
Hence by Carathéodory’s theorem
\[ |\gamma_n| \leq 2\Re e^{i\theta} \leq 2 \quad \text{for} \quad n = 1, 2, \ldots. \]
Consequently
\[ G(z) \ll (1 + z)/(1 - z). \]
On the other hand from (3.11) we have
\[ \psi(z) = ie^{-i\alpha}(-1)^p \exp \left( \frac{i}{2} \sum_{s=1}^{2p} \theta_s \right) \cdot z^p G(z) / \left\{ (1 - z^2) \prod_{s=1, i}^{2p} (e^{i\theta_s} - z) \right\} \]
which is dominated by
\[ z^p \left( \frac{1 + z}{1 - z} \right) \cdot \frac{1}{1 - z^2} \cdot \frac{1}{(1 - z)^{2p - 2}} = \frac{z^p}{(1 - z)^{3p}} \]
since we have (3.12).

From (3.4) and (3.5), we have
\[ f(z) = \psi(z) \prod_{i=1}^{p-q} (z - \beta_i)(1 - \bar{\beta}_i z) / \left( \prod_{i=1}^{p-q} \beta_i z^{p-q} \right) \]
which is dominated by
\[ z^p \prod_{i=1}^{p-q} \left( 1 + \left| \frac{z}{\beta_i} \right| \right) \left( 1 + \left| \beta_i \right| z \right) \cdot \frac{1}{z^{p-q}} = F(z) \]
since we have (3.13). Hence we obtain
\[ \left| a_n \right| \leq B_n, \quad n = q + 1, q + 2, \ldots, \]
and
\[ \left| f(re^{i\theta}) \right| \leq F(r) \quad \text{for } r < 1. \quad \text{q.e.d.} \]

4. A class of functions related to $D(p)$.

**Definition 3.** Let $w = f(z)$ be regular for $|z| \leq 1$ and $C$ be the image curve of $|z| = 1$. Let, further, $P$ be the orthogonal projection of $f(e^{i\theta})$ onto a straight line. Then $P$ will move on the straight line both positively or negatively when $\theta$ varies from 0 to $2\pi$. If $P$ changes its direction of movement $2p$ times when $\theta$ varies from 0 to $2\pi$, then $f(z)$ is said to be convex of order $p$ in the direction which is perpendicular to the straight line. This class of functions has recently been studied by M. S. Robertson [9].

Especially if, when we represent $f(z)$, $zf'(z)$ in the same plane, the straight line is parallel to a diametral line of $zf'(z)$, then $f(z)$ is said to be a member of $F(p)$.

**Lemma 3.** $f(z)$ is a member of the class $F(p)$ if and only if $zf'(z)$ belongs to the class $D(p)$. 
PROOF. This is a generalization of M. S. Robertson’s lemma [6].

It is sufficient to prove the lemma in the case where the diametral line of \( f(z) \) is the real axis, since in the other cases we may consider \( e^{i\alpha f(z)} \) with a suitable choice for the real parameter \( \alpha \).

Using the identity
\[
\Im \{z f'(z)\} = - \partial \Re f(z)/\partial \theta \quad \text{for } |z| = 1
\]
we see, under the hypothesis,
\[
\begin{align*}
\Im \{z f'(z)\} &= - \partial \Re (e^{i\theta})/\partial \theta > 0 \quad \text{for } \theta_{2s-1} < \theta < \theta_{2s}, \\
\Im \{z f'(z)\} &= - \partial \Re (e^{i\theta})/\partial \theta < 0 \quad \text{for } \theta_{2s} < \theta < \theta_{2s+1},
\end{align*}
\]
with \( s = 1, 2, \ldots, \rho, \theta_i = \theta_1 + \pi, \theta_{2\rho+1} = \theta_1 + 2\pi \).

Hence \( f(z) \in F(p) \) if and only if \( z f'(z) \in D(p) \).

**Theorem 2.** Let
\[
(4.1) \quad f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n
\]
be a function of the set \( F(p) \). Suppose that in addition to the \( (q-1) \)th order critical points at \( z = 0 \), the function \( f(z) \) has exactly \( p-q \) critical points \( \alpha_1, \alpha_2, \ldots, \alpha_{p-q} \) such that \( 0 < |\alpha_j| < 1, j = 1, 2, \ldots, p-q \). Then
\[
(4.2) \quad |a_n| \leq q C_n/n, \quad n = q + 1, q + 2, \ldots,
\]
\[
(4.3) \quad |f(re^{i\theta})| \leq q \int_0^r \frac{F(r)}{r} \, dr \quad \text{for } r < 1,
\]
\[
(4.4) \quad |f'(re^{i\theta})| \leq q F(r)/r, \quad \text{for } r < 1,
\]
where \( C_n \) and \( F(r) \) are defined by
\[
F(z) = \frac{z^q}{(1 - z)^{2p}} \prod_{j=1}^{p} \left( 1 + \frac{z}{|\beta_j|} \right) \left( 1 + z |\beta_j| \right) \quad \text{for } z \in D(p)
\]
\[
(4.5) \quad f(z) = z^q + \sum_{n=q+1}^{\infty} C_n z^n.
\]

**Proof.** Since \( f(z) \in F(p) \),
\[
\frac{1}{q} z f'(z) = z^q + \sum_{n=q+1}^{\infty} n a_n z^n \in D(p)
\]
by Lemma 3.

By using the main theorem we have (4.2) and (4.4). By integrating \( f'(z) \) along a radius we have, for \( z = re^{i\theta} \),
\[ |f(re^{\theta})| = \left| \int_0^r f'(z)dz \right| \leq \int_0^r |f'(re^{\theta})| dr \leq q \int_0^r \frac{F(r)}{r} dr \]

for \( r < 1 \),

which completes the proof.

5. Subclasses of \( D(p) \) and \( F(p) \).

**Corollary 1.** Let \( f(z) \) in the form (3.1) be regular for \( |z| \leq 1 \) and assigned with the same zeros as in Theorem 1. Suppose that \( f(z) \) satisfies one of the following conditions:

(i) \( \Re\left[zf'(z)/f(z)\right] > 0 \) for \( |z| = 1 \),

(ii) \( f(1) = \text{real}, \ f(-1) = \text{real} \) and \( \Im f(e^{\theta}) \) changes sign \( 2p \) times on \( \theta = 1 \),

(iii) \( f(z) \in T(p) \).

Then (3.2) and (3.3) hold.

**Proof.** (i) Since there exists at least one diametral line of \( f(z) \) by Lemma 1', and since \( f(z) \) is starlike of order \( p \) in every direction by the fact that \( \Re\left[zf'(z)/f(z)\right] > 0 \) on \( |z| = 1 \) and \( f(z) \) has \( p \) zeros in \( |z| < 1 \), \( f(z) \) is evidently starlike of order \( p \) in the direction of the above diametral line.

(ii) In this case the diametral line of \( f(z) \) is evidently the real axis and is starlike of order \( p \) in this direction by our hypothesis, which proves the corollary by using the main theorem.

(iii) This is a direct consequence of the preceding (ii).

**Corollary 2.** Let \( f(z) \) in the form (4.1) be regular for \( |z| \leq 1 \) and assigned with the same critical points as in Theorem 2. Suppose that \( f(z) \) satisfies one of the following conditions:

(i) \( 1 + \Re\left[zf''(z)/f'(z)\right] > 0 \) for \( |z| = 1 \).

(ii) \( f'(1) = \text{real}, \ f'(-1) = \text{real}, \) and \( f(z) \) is convex of order \( p \) in the direction of the imaginary axis.

(iii) In (4.1) the coefficients are all real and \( f(z) \) is convex of order \( p \) in the direction of the imaginary axis.

Then (4.2), (4.3), and (4.4) hold.

**Proof.** (i) By our hypothesis \( zf'(z) \) has \( p \) zeros in \( |z| < 1 \) and \( \Re\left[z\{zf''(z)\}'/\{zf'(z)\}\right] > 0 \) on \( |z| = 1 \). Hence \( zf'(z) \) is starlike of order \( p \) in every direction. Consequently \( zf'(z) \in D(p) \) by Corollary 1 adopting (i). Accordingly \( f(z) \in F(p) \) by Lemma 3.

(ii) By our hypothesis \( -\partial\Re f(z)/\partial \theta \) changes sign \( 2p \) times on \( |z| = 1 \). Accordingly \( \Im\left\{zf'(z)\right\} \) changes sign \( 2p \) times on \( |z| = 1 \) by
Lemma 3. And \( f'(1) = \text{real} \), \( (-1)f'(-1) = \text{real} \). Hence \( zf''(z) \in D(p) \).
Consequently \( f(z) \in F(p) \).

(iii) This is a special case of (ii).

REFERENCES


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