1. **An isomorphism theorem.** In this paper we discuss a particular method of introducing a convolution product into certain topological linear spaces so that they become algebras. The space in each case will be essentially the dual space of a linear space of continuous functions, and arises in the study of general group algebras and in the theory of distributions.

Let $G$ be a topological group, written additively but not necessarily abelian. Let $C^*(G)$ be the linear space of all complex-valued continuous functions on $G$. For $x \in G$ let $U_x$ be the left translation operator which sends a function $\phi$ into the function $U_x \phi = \phi(x+t)$, whose value at $t$ is $\phi(x+t)$. Let $X$ be a subspace of $C^*(G)$, topologized by a locally convex topology $\tau$. Let $L(X)$ be the space of linear functionals on $X$ which are $\tau$ continuous on all $\tau$ bounded sets of $X$. Let $B(X)$ be the algebra of linear operators on $X$ which are $\tau$ bounded and $\tau$ continuous on the $\tau$ bounded sets of $X$. These contain the usual dual space $X'$ and endomorphism algebra $E(X)$ and in certain important cases coincide with them, for example if $\tau$ is a norm topology. The operators $U_x$ form a group $G_0$ of operators on $C^*(G)$ anti-isomorphic to $G$. If $X$ is invariant under $G_0$ and $\tau$ suitable, we may regard $G_0$ as part of $B(X)$. In this case, let $B^0(X)$ be the set of all $T$ in $B(X)$ which commute with $G_0$. $B^0(X)$ is evidently a subalgebra of $B(X)$.

Our first theorem shows that under certain conditions $L(X)$ and $B^0(X)$ are isomorphic as linear spaces, thus permitting us to introduce an algebraic structure into $L(X)$.

We first define two fundamental mappings between operators and functionals. For any operator $T$ in $B(X)$, let $F$ be a functional on $X$ defined by

$$ F(\phi) = T(\phi)(0). $$

For a functional $F$ in $L(X)$, let $T$ be an operator on $X$ defined by

$$ T(\phi) = \psi \quad \text{where} \quad \psi(x) = F(U_x \phi). $$

**Theorem 1.** If the functional $F$ defined by (1) is in $L(X)$ for each $T$ in $B^0(X)$, and if $T$ defined by (2) is in $B(X)$ for each $F$ in $L(X)$, then (1) (or (2)) defines an isomorphism of $B^0(X)$ and $L(X)$ as linear spaces.

Presented to the Society, April 27, 1951; received by the editors April 3, 1951 and, in revised form, January 11, 1952.
We need show only that (1) and (2) are inverse, and that $T$ defined in (2) lies in the subalgebra $B^0(X)$. For the latter, $T(U_x\psi)(x) = F(U_x U_y \phi) = F(U_y - U_x \phi) = \psi(y + x) = U_y(\psi)(x) = U_x T(\phi)(x)$. Since this holds for all $x$ and $\phi$, $T(U_x U_y \phi) = U_x T(\phi)(x) = U_y(\psi)(x) = U_x T(\phi)(x)$, and $T' = T$. Thus, (1) is a one-to-one embedding of $B^0(X)$ in $L(X)$. Given $F$ in $L(X)$, define $T$ by (2), and $T'$ by (1). For any $\phi$, $F'(\phi) = T(\phi)(0) = F(U_x \phi) = F(\phi)$, and $F' = F$.

Since $B^0(X)$ is also an algebra, the operation of operator multiplication induces a multiplication of functionals in the space $L(X)$. Given two functionals $F_1$ and $F_2$, let $T_1$ and $T_2$ be their images in $B^0(X)$. Then, $F_1 \cdot F_2$ is defined to be the functional corresponding to $T_1 T_2$. Under $\cdot$, $L(X)$ becomes an algebra isomorphic to $B^0(X)$. The identity operator $I$ corresponds to the functional $\delta$ defined by $\delta(\phi) = I(\phi)(0) = \phi(0)$; this "delta function" is then the unit for $L(X)$. The operation $\cdot$ can be given a more explicit description. For the value of $F$ at $\phi$, let us introduce the alternate notation $F(\phi(t) dt)$. Then, the function $\psi$ of (2) may be given by $\psi(x) = F(\phi(x + t) dt)$ and $(F_1 \cdot F_2)(\phi)$ is readily seen to be $F_1(F_2(\phi(x + t) dt)) dx$. The $\cdot$ product can thus be considered as a generalization of the familiar convolution of functions and of measures.

The remaining sections will deal with special choices of $X$ and $\tau$ under which the hypotheses of Theorem 1 may be established. Explicitly, it must be shown that $F$ in (1) is $\tau$ continuous on $\tau$ bounded sets, that $\psi$ in (2) lies in $X$, and that the mapping $\phi \mapsto \psi$ is $\tau$ bounded and $\tau$ continuous on $\tau$ bounded sets. In addition, $X$ must be invariant under left translation, and the operators $U_x$ bounded and continuous.

2. Spaces of continuous functions. We distinguish certain special subspaces of $C^*(G)$. The first, $C^0(G)$, is the set of functions $\phi$ having the property that for each, there is an open set $V \subset G$ such that $\phi$ is bounded on $V$ and all of its translates. $C(G)$ is the set of all $\phi$ which are bounded on $G$, and $C_u(G)$ is the set of $\phi$ in $C(G)$ which are uniformly continuous on $G$. On each of these, several familiar topologies may be imposed. We shall use four, labeled as $\omega$, $\kappa$, $\beta$, and $\sigma$. For convenience, set $\|\phi\|_s = \sup_{x \in S} |\phi(x)|$ for $S \subset G$. The topology $\omega$ is that of pointwise convergence on $G$, and is the smallest (weakest) of the topologies. The largest is $\sigma$ which is that of uniform convergence on $G$. The former is applicable to $C^*$ and its subspaces, and the latter to $C$. $\sigma$ is a norm topology, defined by $\|\phi\|_\sigma$ and, under it, $C$ and $C_u$ are Banach spaces. When $G$ is locally compact, two additional topologies will be used. The $\kappa$ topology is the usual one of
compact convergence, and is applicable to $C^* = C^d$, and more especially to $C$. It may be defined by the collection of pseudonorms $\|\phi\|_K$ for compact sets $K \subset G$. It is not complete. The $\beta$ topology, or “strict” topology, may be defined as follows. Let $g$ be any non-negative function in $C(G)$ which vanishes at infinity, and set $\|\phi\|_\sigma = \|\phi g\|_\sigma$. $\beta$ is the convex topology defined by this collection of pseudonorms. It is applicable to $C$ and, as the following lemma shows, it has some of the better features of both $\sigma$ and $\kappa$. It is similar to a topology introduced by Beurling [1].

**Lemma.** The $\beta$ topology on $C(G)$ has the following properties: (i) $\kappa \leq \beta \leq \sigma$, (ii) $\beta$ and $\sigma$ have the same bounded sets, (iii) $\beta$ and $\kappa$ coincide on $\sigma$ bounded sets, (iv) $\beta$ is a sequentially complete topology.

Since $\|\phi\|_\sigma \leq \|\phi\|_\sigma \|g\|_\sigma$, $\beta$ is smaller than $\sigma$. Given a compact set $K \subset G$, choose $g_0$ so that $g_0(x) = 1$ on $K$. Then, $\|\phi\|_K \leq \|\phi\|_{g_0}$ and $\kappa \leq \beta$. Any uniformly bounded set is strictly bounded. Conversely, if $S \subset C(G)$ is strictly bounded but were not uniformly bounded, $\phi_n \in S$ and $x_n \in G$ could be chosen with $|\phi_n(x_n)| = \lambda_n$ and $\lim \lambda_n = \infty$. The sequence $\{x_n\}$ is convergent in $G$ to infinity, and we may choose $g$ so that $g(x_n) = \lambda_n^{-1/2}$. Then, $\|\phi_n\|_\sigma \geq \lambda_n^{1/2}$, contradicting the strict boundedness of $S$. To prove (iii), let $S$ be a subset of $C(G)$ such that $\|\phi\|_\sigma \leq M$ for all $\phi \in S$, and let $\phi_0$ be in the $\kappa$ closure of $S$. Given $\varepsilon > 0$ and any $g$ choose $K$ so that $\|g\|_\sigma \leq \varepsilon$. Then,

$$
\|\phi - \phi_0\|_\sigma \leq \|\phi - \phi_0\|_K \|g\|_\sigma + \|\phi - \phi_0\|_\sigma \|g\|_\sigma - K \\
\leq \|\phi - \phi_0\|_K \|g\|_\sigma + \varepsilon \|\phi - \phi_0\|_\sigma.
$$

If $\phi \in S$ and $\|\phi - \phi_0\|_K < \varepsilon$, then

$$
\|\phi - \phi_0\|_\sigma \leq \varepsilon \|g\|_\sigma + (M + \|\phi_0\|_\sigma) \varepsilon
$$

and $\phi_0$ lies in the strict closure of $S$. Finally, a $\beta$ Cauchy sequence is $\beta$ bounded and hence $\sigma$ bounded, and therefore $\kappa$ and hence $\beta$ convergent to a member of $C(G)$.

It is evident from the above that a sequence $\{\phi_n\}$ is $\beta$ convergent if and only if $\kappa$ convergent and uniformly bounded. Briefly, $\beta$ may be described as the convex topology on $C(G)$ obtained from the pseudotopology of $\kappa$ convergent uniformly bounded sets.

We shall set $\|F\| = \sup \|F(\phi)\|$, taken over all $\phi$ in $X$ with $\|\phi\|_\sigma \leq 1$. Since a uniformly bounded set is bounded in each of the topologies, $\|F\|$ is finite for every $F$ in $L(X)$. Similarly, if $T \in B(X)$, we set $\|T\| = \sup \|T(\phi)\|_\sigma$, taken as above. We cannot immediately conclude that $\|T\| < \infty$, although this follows for $T \in B^b(X)$ from the next theorem.
Theorem 2. Under the convolution product * and the norm || | |, the spaces L(X) become topological algebras in the following cases: (i) \( X = C^\omega(G) \), \( \tau = \omega \), (ii) \( X = C(G) \), \( \tau = \kappa \) or \( \beta \), (iii) \( X = C_\omega(G) \), \( \tau = \sigma \).

We shall use Theorem 1 to establish a norm preserving isomorphism between \( B^0(X) \) and \( L(X) \) in each case. We first observe that each \( X \) is invariant under \( U_x \). Moreover, \( \| U_x \|_s = \| \phi \|_{\tau + s} \) and \( \| U_x \phi \|_\sigma = \| \phi \|_\sigma \) where \( h = U_{-x}g \). Thus, in each case, the operators \( U_x \) lie in \( B(X) \). Assume now that \( T \in B^0(X) \) and define \( F \) by (1); \( F(\phi) = T(\phi)(0) \). Since \( T \) is \( \tau \) continuous on \( \tau \) bounded sets and since \( \omega \leq \tau \), \( F \) is in \( L(X) \). Consider the mapping \( x \to U_x \phi \) of \( G \) into \( X \). If \( \phi \) is unbounded, but lies in \( C^\omega \), there is a neighborhood \( V \) of 0 in \( G \) such that the set of all \( U_x \phi \) for \( x \in V \) is an \( \omega \) bounded subset. Since \( \phi \) itself is continuous, the mapping \( x \to U_x \phi \) of \( G \) into \( X \) is continuous at 0. If \( F \in L(X) \) in case (i), \( F \) is continuous on \( \omega \) bounded sets, and \( \psi(x) = F(U_x \phi) \) is continuous for \( x \in V \). Moreover, \( F \) is bounded on \( \omega \) bounded sets so that \( \psi(x) \) is bounded on \( V \). Similarly, \( \psi \) is bounded and continuous on each translate of \( V \), and thus lies in \( C^\omega \). We turn next to case (ii). Since \( \phi \in X \) is now bounded, the set of all \( U_x \phi \) for \( x \in G \) is a \( \sigma \) bounded set, and thus \( \kappa \) and \( \beta \) bounded. Since \( \kappa \) and \( \beta \) coincide on \( \sigma \) bounded sets, we can discuss these together. If \( K \subset G \) is compact, then \( \phi \) is uniformly continuous on \( K \) and the mapping \( x \to U_x \phi \) is continuous, and since \( |\psi(x)| \leq ||F||_||\phi||_\sigma \), \( \psi \) is bounded and therefore lies in \( C(G) = X \). Finally, in case (iii) \( \phi \) is uniformly continuous on all of \( G \) and \( x \to U_x \phi \) is uniformly \( \sigma \) continuous so that \( \psi(x) \) is uniformly continuous and therefore in \( C_\omega \). In each case, we have verified that (2) defines a linear transformation \( T \) of \( X \) into itself. It remains to show that \( T \) is appropriately continuous and bounded. Since \( ||T(\phi)||_\sigma = ||\phi||_\sigma \leq ||F||_||\phi||_\sigma \) we see that, in all cases, \( ||T|| \leq ||F|| < \infty \). Since \( T \) is then \( \sigma \) continuous on \( \sigma \) bounded sets, the proof of case (iii) is complete. Case (i) is also easily disposed of. For fixed \( x \), \( F(U_x) \) is continuous on \( \omega \) bounded sets, and it then follows that \( T \) is \( \omega \) continuous on \( \omega \) bounded sets. Similarly, since \( F \) must be bounded on \( \omega \) bounded sets, \( T \) carries \( \omega \) bounded sets into \( \omega \) bounded sets. Case (ii) requires slightly more argument. Here \( X = C(G) \) and \( \tau = \kappa \) or \( \beta \).

We discuss \( \kappa \) first. Let \( S \) be a \( \kappa \) bounded set in \( X \) containing 0, and let a compact set \( K \subset G \) and \( \epsilon > 0 \) be given. Let \( S^* = \{ x \in K, \phi \in S \} \). \( S^* \) is also \( \kappa \) bounded. Since \( F \in L(X) \) is \( \kappa \) continuous on \( S^* \), choose \( K_1 \) and \( \delta \) so that \( \phi \in S^* \) and \( ||\phi||_{K_1} < \delta \) imply \( ||F(\phi)|| < \epsilon \). Set \( K_2 = K + K_1 \). Then, if \( x \in K_2 \), \( ||U_x \phi||_{K_1} \leq ||\phi||_{K_1} \) so that \( \phi \in S \) and \( ||\phi||_{K_2} < \delta \) imply that \( ||F(U_x \phi)|| < \epsilon \) for all \( x \in K \). Rephrased, this states that if \( \phi \in S \) and \( ||\phi||_{K_1} < \delta \), then \( ||\psi||_{K} < \epsilon \), proving that \( T \) is
\( k \) continuous on \( k \) bounded sets. We have also proved that the image of \( S \) is bounded, since \( |F(\phi)| \leq A \) holds for all \( \phi \in S^* \), so that if \( \phi \in S \), \( |\psi|_\kappa \leq A \). The argument for \( \tau = \beta \) is implicit in the discussion above, since the set \( S \) being \( \beta \) bounded is \( \sigma \) bounded, and so is \( S^* \), and \( \beta \) and \( \kappa \) agree on such sets. Finally, to show that the correspondence established in (1) and (2) is norm preserving, we note that we have shown above that \( |T| \leq ||F|| \). Since \( |F(\phi)| = |T(\phi)(0)| \leq \|T(\phi)\|_\sigma \), we also have \( ||F|| \leq ||T|| \).

It perhaps should be pointed out that the space \( L(X) \) in case (ii) contains an image of the usual \( L^1 \) algebra of \( G \). If \( f \in L^1(G) \), then the functional \( F \) defined by \( F(\phi) = \int f \phi \) is in \( L(C(G)) \) if \( \tau \) is chosen as \( \beta \), although not if \( \tau = \kappa \). This approach, using case (iii), has been discussed elsewhere [2].

3. The space of distributions. Let \( G \) be the additive group of reals, with the usual locally compact topology. Let \( \mathcal{E} \) be the subspace of \( C^*(G) \) composed of the infinitely differentiable functions, and let \( D \) be the subspace of those \( \phi \) which have compact support, that is, vanish off a compact set. On \( \mathcal{E} \), we impose the topology \( \tau \) generated by the pseudonorms \( \|\phi\|_{\kappa,p} = \|D^p\phi\|_\kappa \) where \( D \) is the differentiation operator, and \( K \) is a compact set. Clearly \( \tau \)-lim \( \phi_n = 0 \) means merely that \( \kappa \)-lim \( D^p\phi_n = 0 \) for \( p = 0, 1, \ldots \). Let \( \mathcal{E}' \) and \( E(\mathcal{E}) \) be the usual dual space and endomorphism algebra of \( \mathcal{E} \); \( E^0(\mathcal{E}) \) is again the subalgebra of those \( T \) in \( E(\mathcal{E}) \) which commute with the translation operators \( U_z \).

Theorem 3. \( \mathcal{E}' \) and \( E^0(\mathcal{E}) \) are isomorphic under the mappings (1) and (2).

We first observe that if \( T \) is in \( E^0(C^*) \) where \( C^* \) has the \( \kappa \) topology, and if \( T \) leaves \( \mathcal{E} \) invariant and commutes with \( D \), \( T \) is in \( E^0(\mathcal{E}) \). As before, since \( \omega \leq \tau \), the functional \( F \) defined by (1) is continuous and hence in \( \mathcal{E}' \). Given \( F \) in \( \mathcal{E}' \) and defining \( T \) by (2), \( T \) is easily seen to belong to \( E^0(C^*) \). With \( \psi(x) = F(U_x\phi) \), we have \( D^p\psi(x) = \lim_{h \to 0} F(h^{-1}(U_{x+h} - U_x)\phi) \). For each \( p \), \( D^p\phi \) is continuous, and hence uniformly continuous on compact sets, and thus \( \tau \)-lim \( h^{-1}(U_{x+h} - U_x)\phi = U_x D\phi \). Since \( F \) is \( \tau \) continuous, \( D\psi(x) = F(U_x D\phi) \), and by induction, \( D^p\psi(x) = F(U_x D^p\phi) \). This proves both that \( \psi \) lies in \( \mathcal{E} \) so that \( T \) leaves \( \mathcal{E} \) invariant, and that \( T \) commutes with \( D \). By the remark above, \( T \) is in \( E^0(\mathcal{E}) \).

The algebra \( E^0(\mathcal{E}) \) may be considered as the set of all \( \kappa \) continuous operators on \( C^* \) which carry \( \mathcal{E} \) into itself and commute with \( D \) and with translations. It contains the subalgebra of all differential operators with constant coefficients and of finite order, but does not
contain such operators as $x^2 D$. It also contains such integral transforms as $T(\phi) = \psi$ where

$$\psi(x) = \int_{-\infty}^{\infty} H(x - t)\phi(t)\,dt$$

where $H \in \mathcal{D}$. More generally, the theorem asserts that the general operator $T$ has the representation $T(\phi) = \psi$ with $\psi(x) = F(U_x \phi)$ for a unique $F$ in $\mathcal{E}'$. This latter space is exactly the space of distributions having compact support (see [3, chap. 3]). As before, the multiplication of operators in $\mathcal{E}'$ induces a convolution product into $\mathcal{E}'$ so that it too becomes an algebra. It is readily seen that this is the same product as that introduced in a different way by Schwartz. That portion of the theory of distributions relating to $\mathcal{E}'$ can be transferred with some advantage to the operator algebra $\mathcal{E}'$. In place of the “derivative” of a distribution, the operator $D$ itself is present; if $F$ is the distribution corresponding to the operator $T$, the derivative $F'$ corresponds to the operator $-DT$.

Turning now to the space $\mathcal{D}$, the importance of this subspace of $\mathcal{E}$ lies partly in the fact that, on it, $D$ is a one-to-one operator, whose range is the set of all $\phi$ with $\int_{-\infty}^{\infty} \phi = 0$. We can consider $\mathcal{D}$ under the relative $\tau$ topology or under the Schwartz topology $\tau'$ in which a convergent sequence must have a common supporting set. The dual space $\mathcal{E}'$ is the space of all distributions; let $\mathcal{E}(\mathcal{D})$ be the corresponding endomorphism algebra. The natural mappings (1) and (2) do not set up an isomorphism of $\mathcal{D}'$ and $\mathcal{E}(\mathcal{D})$; instead, $\mathcal{E}(\mathcal{D})$ is embedded isomorphically in a proper part of $\mathcal{D}'$. For example, the distribution 1 corresponds to the functional $F$ where $F(\phi) = \int_{-\infty}^{\infty} \phi$. The corresponding operator defined by (2) sends $\phi$ into $\psi$, with $\psi(x) = F(\phi)$ for all $x$; $\psi$ is a constant function, and lies in $\mathcal{E}$ but not $\mathcal{D}$. It is possible to obtain a space of operators which is isomorphic to the whole space $\mathcal{D}'$. Consider $\mathcal{E}(\mathcal{D}; \mathcal{E})$, the set of all continuous linear transformations of $\mathcal{D}$ into $\mathcal{E}$ which commute with $U_x$ for $x \in G$. The previous type of argument shows that (1) and (2) set up an isomorphism between $\mathcal{D}'$ and $\mathcal{E}(\mathcal{D}; \mathcal{E})$. The convolution of distributions can now be discussed in terms of the ordinary product of operators. Given $T_1$ and $T_2$ in $\mathcal{E}(\mathcal{D}; \mathcal{E})$, the product $T_1 T_2$ is an operator whose domain is $\mathcal{D} \cap T^{-1}_2(\mathcal{D})$. It corresponds to a distribution only if this domain is $\mathcal{D}$ (and thus if $T_2(\mathcal{D}) \subseteq \mathcal{D}$) or, more generally, if the domain of $T_1$ can be expanded to include the range of $T_2$. For example, we recall that a distribution with compact support corresponds to an operator $T$ mapping $\mathcal{E}$ into $\mathcal{E}$, and $\mathcal{D}$ into $\mathcal{D}$. It is therefore possible to form the convolution of an arbitrary distribution and one with compact sup-
port, in either order. (See [3, chap. 6].) In going from an operator to the corresponding distribution, only the action of the operator on $\mathcal{D}$ is considered; in this contraction, some properties of the operator algebra are lost, in particular convolution of distributions fails to be associative, although the product of operators of course is [3, chap. 6, §5]. This suggests that it would perhaps be fruitful to study the general algebra of operators $T$ whose domains are subspaces of $\mathcal{E}$ containing $\mathcal{D}$, and whose ranges lie in $\mathcal{E}$.

References


University of Wisconsin