ON THE FACTORIZATION OF SQUAREFREE INTEGERS

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In recent years several papers [1; 3; 4; 5; 6; 7; 9; 10; 11] have appeared dealing with the problem of "Factorisatio numerorum," the number \( f(n) \) of representations of an integer \( n \) as an ordered product of factors greater than 1. As a result, the basic combinatorial properties of \( f(n) \) and the asymptotic behavior of its summatory function are well known. In this paper, I determine the asymptotic behavior of \( f(n) \) itself for squarefree \( n \) and use the result to determine a "normal" order of \( f(n) \) for all \( n \).

If \( n \) is written as a product of powers of distinct primes \( \rho \), then \( f(n) \) is evidently a symmetric function of the exponents of the \( \rho \)'s. If \( n \) is squarefree, all the exponents are 1 and \( f(n) \) can be considered a function of the single variable \( r \), the number of distinct prime factors of \( n \). It is therefore convenient to define, for positive integral \( r \):

\[
h(r) = f(S_r),
\]

where \( S_r \) represents any squarefree integer with \( r \) prime factors.

A (convergent) asymptotic expansion of \( h(r) \) is given by:

**Theorem 1.** (a) For every positive integral \( r \),

\[
h(r) = 2^{-1} r! \left( \log 2 \right)^{-1} \{ 1 + R_r \},
\]

and the remainder \( R_r \) can be expressed in the form:

\[
2 \sum_{n=1}^{\infty} (\cos \theta_n)^{r+1} \cos \{ (r + 1)\theta_n \},
\]

where \( \theta_n \) is defined by:

\[
\cos \theta_n = \frac{\log 2}{2\pi n} \left\{ 1 + \left( \frac{\log 2}{2\pi n} \right)^2 \right\}^{-1/2}.
\]

(b) For every positive integral \( r \),

\[
| R_r | \leq 2 \zeta(r + 1) \left( \frac{\log 2}{2\pi} \right)^{r+1}
\]

where \( \zeta(r + 1) \) is the Riemann zeta function.

**Proof.** Sen [10] proves the identity:\(^1\)

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\(^1\) The identity (5) is a special case of the general relation \( f(n) = 2^{-1} \sum d_k(n) 2^{-k} \), which relation does not seem to appear explicitly anywhere in the literature. \( d_k(n) \) is the number of ordered factorizations of \( n \) into \( k \) factors greater than or equal to 1.
Part (a) of Theorem 1 is merely the result of applying the Euler-MacLaurin formula to (5).

For this purpose, let \( g_r(x) = 2^{-x^2-x} \) for \( x \) real. Then \( g_r(x) \) and all its derivatives vanish at \(+\infty\), are integrable from \(0\) to \(+\infty\), and \( g_r(x) \) and its derivatives up to order \( r-1 \) vanish at \( x=0 \). Since \( g_r(n) = 2^{-n^2-n} \), the Euler-MacLaurin formula gives for every positive integral \( k \):

\[
h(r) = \int_0^\infty g_r(x)dx - \sum_{i=1}^k \frac{B_{2i}}{(2i)!} g_r^{(2i-1)}(0) + \int_0^\infty p_{2k+1}(x)g_r^{(2k+1)}(x)dx,
\]

the notation being that of Knopp \([8, pp. 524-526]\).

The first term on the right gives the dominant term in (1), since

\[
\int_0^\infty g_r(x)dx = \frac{1}{2} \int_0^\infty x^r 2^{-x}dx = \frac{1}{2} r! (\log 2)^{-r-1}.
\]

Set

\[
I_{r,k} = \int_0^\infty p_{2k+1}(x)g_r^{(2k+1)}(x)dx.
\]

Then

\[
|I_{r,k}| \leq 4(2\pi)^{-2k-1} \int_0^\infty |g_r^{(2k+1)}(x)| \, dx
\]

since \( |P_{2k-1}(x)| \leq 4(2\pi)^{-2k-1} \) \([8, p. 527]\).

From Leibnitz's rule, for any positive integral \( m \):

\[
g_r^{(m)}(x) = \frac{1}{2} \sum_{i=0}^{\min(m,r)} (-1)^{m-i} C_m, i \, q!, (\log 2)^{m-i} x^r i! 2^{-x}.
\]

It immediately follows that:

\[
|g_r^{(2k+1)}| \leq \frac{1}{2} \sum_{i=0}^{\min(m,r)} C_{2k+1}, i \, q!, (\log 2)^{2k+1-i} x^r i! 2^{-x}.
\]

If \( f_i(n) \) be defined as the number of factorizations of \( n \) into \( j \) factors greater than \( n \), then the general relation is a consequence of the identity (Strehler \([11]\)) \( d_k(n) = \sum_{j=1}^n C_{k,j} f_j(n) \).
When this is substituted into the inequality for $|I_{r,k}|$, and the integral evaluated, the result is:

$$|I_{r,k}| \leq \frac{1}{\pi} \frac{r!}{(\log 2)^r} \left(\frac{\log 2}{2\pi}\right)^{2k} \sum_{i=0}^{2k+1} C_{2k+1,i}$$

$$= \frac{2}{\pi} \frac{r!}{(\log 2)^r} \left(\frac{\log 2}{\pi}\right)^{2k}$$

where the expression on the right tends to zero, for fixed $r$, as $k$ increases.

Since the remainder in (6) tends to zero as $k$ increases, (6) may be written:

$$h(r) = \frac{1}{2} r! (\log 2)^{-r-1} - \sum_{i=1}^{\infty} \frac{B_{2i}}{(2j)!} g_{r-i}(0).$$

From (7),

$$g_{r-i}(0) = \frac{1}{2} (-1)^{2j-r-1} C_{2j-1,r} (\log 2)^{2j-1}. $$

Hence,

$$R_r = (-1)^r \sum_{i=1}^{\infty} \frac{B_{2i}}{(2j)!} C_{2j-1,r} (\log 2)^{2j}.$$ 

Now [8, p. 237],

$$\frac{B_{2i}}{(2j)!} = (-1)^{i-1} \frac{2}{(2\pi)^{2i}} \sum_{n=1}^{\infty} \frac{1}{n^{2i}} .$$

Therefore,

$$R_r = (-1)^{r+1} \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} 2(-1)^i C_{2j-1,r} \left(\frac{\log 2}{2\pi}\right)^{2i}$$

$$= (-1)^{r+1} \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} 2 \cos \frac{\pi j}{2} C_{j-1,r} a_n i$$

where $a_n = (\log 2)/2\pi$. Substitution of $i^j + (-i)^j$ for $2 \cos (2^{-1}\pi j)$, interchanging of the order of summation, and use of the identity

$$\sum_{i=1}^{\infty} C_{j-1,r} y^i = y^{r+1}(1 - y)^{-r-1}$$

leads to the expression:
\[ R_r = (-1)^r \sum_{n=1}^{\infty} a_n \left\{ (-a_n)^{-r-1} + (i - a_n)^{-r-1} \right\} \]

which, when put back into real form, is (2).

Part (b) of Theorem 1 follows from (2) and (3), for:

\[ \left| R_r \right| \leq 2 \sum_{n=1}^{\infty} \left| \cos \theta_n \right|^{r+1} \]
\[ \leq 2 \sum_{n=1}^{\infty} \left( \frac{\log 2}{2\pi n} \right)^{r+1} = 2 \left( \frac{\log 2}{2\pi} \right)^{r+1} \zeta(r+1). \]

Deviating slightly from Hardy and Ramanujan [2], I define a "normal" order of a numerical function \( F(n) \) as a function \( G(n) \) such that \( F(n) \sim G(n) \) as \( n \) increases over a set that includes almost all integers ("almost all" in the sense of Hardy and Ramanujan [2]). Under this definition, a consequence of Theorem 1 is:

**Theorem 2.** A normal order of \( \log f(n) \) is \( \log \log n \log \log \log n \).

**Proof.** Let \( r_1 = r_1(n) \) be the number of distinct prime factors of \( n \), and \( r_2 = r_2(n) \) be the total number of prime factors of \( n \), distinct or not. Evidently, \( \log h(r_1) \leq \log f(n) \leq \log h(r_2) \). By Theorem 1, \( \log h(r) \sim r \log r \left\{ 1 + o(1) \right\} \) as \( r \) increases. Hence,

\[ r_1 \log r_1 \left\{ 1 + o(1) \right\} \leq \log f(n) \leq r_2 \log r_2 \left\{ 1 + o(1) \right\}. \]

Now \( r_1 \) and \( r_2 \) both have the normal orders \( \log \log n \) [2, Theorems B' and C']. Therefore, for almost all \( n \), \( \log h(r_1) \leq \log f(n) \leq \log h(r_2) \). By the prime-number theorem \( k \log k \sim \log f(n) \) by Theorem 1, \( \log h(k) \sim k \log k \) by Theorem 1. Hence,

\[ \log f(nk) \sim \log f(nk) \sim \log \log n \log \log \log n \left\{ 1 + o(1) \right\} \]

which is equivalent to Theorem 2.

A maximum order of a non-negative numerical function \( F(n) \) defined over an infinite set of positive integers may be defined as a function \( G(n) \) such that \( \limsup_{n \to \infty} F(n)/G(n) = 1 \).

**Theorem 3.** A maximum order of \( \log f(n) \) for squarefree \( n \) is \( \log n \).

**Proof.** Take \( n_k \) to be the product of the first \( k \) primes. Then \( \log f(n_k) = \log h(k) \) and \( \log f(n_k) \sim k \log k \), since \( \log h(k) \sim k \log k \) by Theorem 1. By the prime-number theorem \( k \log k \sim \log n_k \), hence

\[ \log f(n_k) \sim \log n_k. \]

For any squarefree integer \( n \), there is an \( n_k \) for which \( f(n_k) = f(n) \), \( n_k \leq n \). Hence \( \log f(n) = \log f(n_k) = \log n_k \left\{ 1 + o(1) \right\} \leq \log n \left\{ 1 + o(1) \right\} \).

Theorem 3 may be compared with the fact (Hille [3], Ikehara...
that the maximum order of $\log f(n)$ for all $n$ is $\rho \log n$, where $\rho$ is the unique positive number with the property $\zeta(\rho) = 2$.

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REFERENCES


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