A GENERALIZATION OF A THEOREM
BY HARDY AND LITTLEWOOD

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1. Suppose that \( \phi(x) \) is non-negative and integrable in \((0, 1)\), so
that it is measurable and finite almost everywhere. If \( M(y) \) is the
measure of the set in which \( \phi(x) \geq y \), \( M(y) \) is a decreasing function of
\( y \). The inverse \( \phi \) of \( M \) is defined by

\[
\phi(M(y)) = y,
\]

and \( \phi(x) \) is a decreasing function of \( x \) defined uniquely in \((0, 1)\)
except for at most an enumerable set of values of \( x \), viz., those cor-
responding to intervals of constancy of \( M(y) \). We may complete the
definition of \( \phi(x) \) by agreeing, for example, that

\[
\phi(x) = \frac{\phi(x - 0) + \phi(x + 0)}{2}
\]
at a point of discontinuity.

We call \( \phi(x) \) the rearrangement of \( \phi(x) \) in decreasing order.

2. The following theorem which is important for its function-
theoretic applications is due to Hardy and Littlewood [1]. The
theorem may be stated in two equivalent forms.

**Theorem A.** Suppose that \( f(x) \) is non-negative and integrable in
\((0, 1)\), that

\[
\Theta(x) = \Theta(x, f) = \max_{0 \leq t < x} \frac{1}{x - t} \int_t^x f(t)dt.
\]

Then

\[
\bar{\Theta}(x) \leq \frac{1}{x} \int_0^x \bar{f}(t)dt
\]

for \( 0 < x \leq 1 \).

**Theorem B.** Suppose that \( f(x) \) satisfies the conditions of Theorem A,
and that \( s(y) \) is any increasing function of \( y \) defined for \( y \geq 0 \). Then

\[
\int_0^1 s\{\Theta(x)\}dx \leq \int_0^1 s\left\{\frac{1}{x} \int_0^x \bar{f}(t)dt\right\}dx.
\]

In this note I generalize the definition of \( \Theta(x) \), and prove theorems
analogous to those above.

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3. Suppose that \( k(t) \) is a non-negative, decreasing function of \( t \),
   defined and integrable in \((0, 1)\).

**Theorem 1.** Suppose that \( f(x) \) is non-negative and integrable in
   \((0, 1)\), that \( k(t) \) satisfies the conditions of the last paragraph,
   that \( k(t)f(t) \) is integrable in \((0, 1)\), and that

\[
(1) \quad \Theta(x) = \Theta(x, f) = \max_{0 < h \leq x} \frac{1}{h} \int_0^h k \left( \frac{t}{h} \right) f(x - h + t) dt.
\]

Then

\[
\Theta(x) \leq \frac{1}{x} \int_0^x k \left( \frac{t}{x} \right) \tilde{f}(t) dt
\]

for \( 0 < x \leq 1 \).

**Theorem 2.** Suppose that \( f(x) \) satisfies the conditions of Theorem 1,
   and that \( s(y) \) is any increasing function of \( y \) defined for \( y \geq 0 \). Then

\[
\int_0^1 s\{\Theta(x)\} dx \leq \int_0^1 s \left\{ \frac{1}{x} \int_0^x k \left( \frac{t}{x} \right) \tilde{f}(t) dt \right\} dx.
\]

The equivalence of Theorems 1 and 2 may be proved in the same
   way as is used to prove the equivalence of Theorems A and B
   \([2, 10.18]\).

It is easily seen that Theorems A and B are the particular case of
   Theorems 1 and 2 where \( k(t) \) is the kernel of the first Cesaro mean,
   \((C, 1)\). The latter theorems will also deal with the case \((C, \delta),
   0 < \delta \leq 1\).

In this paper I prove Theorem 2, from which Theorem 1 may be
   deduced. Firstly, we shall prove Theorem 2 for the special case where

\[
(2) \quad s(t) = s_\alpha(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \alpha, \\ 1 & \text{for } t > \alpha. \end{cases}
\]

In this case, if we rewrite Theorem 2, it is required to prove that

\[
(3) \quad m_{E_{0 \leq x \leq 1}} \left[ x, \Theta(x) > \alpha \right] \leq m_{E_{0 \leq x \leq 1}} \left[ x, \frac{1}{x} \int_0^x k \left( \frac{t}{x} \right) \tilde{f}(t) dt > \alpha \right].
\]

We require the following lemmas:

**Lemma 1.** Suppose that each point of a set in \((a, b)\) is the right-hand
   end point of one or more intervals \((x - h, x)\) of a family \( H \). Then there
   is a finite nonoverlapping set \( S \) of intervals of \( H \) which includes a subset
   \( E' \) of \( E \) such that \( m_{E'_{0 \leq x \leq 1}} > m_{E_{0 \leq x \leq 1}} - \varepsilon \).
This result is due to Sierpinski [3].

**Lemma 2.** If \( f(x) \) is non-negative and integrable in \((0, a+b)\), \( k(t) \) is a non-negative, decreasing function of \( t \) in \((0,1)\), \( k(t)f(t) \) is integrable, and if

\[
\frac{1}{a} \int_0^a k\left(\frac{t}{a}\right)f(t)dt > \alpha, \quad \frac{1}{b} \int_0^b k\left(\frac{t}{b}\right)f(t+a)dt > \alpha,
\]

then

\[
\frac{1}{a+b} \int_0^{a+b} k\left(\frac{t}{a+b}\right)f(t)dt > \alpha.
\]

Since \( k(t) \) is decreasing and integrable in \((0,1)\), it has a non-negative derivative almost everywhere. Also, writing

\[
F(x) = \int_0^x f(t)dt,
\]

and from (4), integrating by parts, we have

\[
k(1)F(a) + \int_0^1 k'(t)F(at)dt > a\alpha,
\]

\[
k(1)\{F(a+b) - F(a)\} + \int_0^1 k'(t)\{F(a + bt) - F(a)\} dt > b\alpha.
\]

Adding, we have

\[
k(1)F(a+b) + \int_0^1 k'(t)\{F(at) + F(a + bt) - F(a)\} dt > (a + b)\alpha.
\]

Similarly, we have

\[
\int_0^{a+b} k\left(\frac{t}{a+b}\right)f(t)dt
\]

\[
= k(1) \int_0^{a+b} f(t)dt + \int_0^1 k'(t) \left\{ \int_0^{(a+b)t} \tilde{f}(s)ds \right\} dt
\]

\[
= k(1)F(a+b) + \int_0^1 k'(t) \left\{ \int_0^{(a+b)t} \tilde{f}(s)ds \right\} dt.
\]

Now it is evident from the definition of \( \tilde{f}(t) \) that, when \( mE = (a+b)t \),

\[
\int_0^{(a+b)t} \tilde{f}(s)ds \geq \int_m^Ef(s)ds.
\]
Hence,

\[ \int_0^{(a+b)t} \tilde{f}(s) ds \geq F(at) + F(a + bt) - F(a), \]

and, since almost everywhere \( k'(t) \geq 0 \), we have

\[ \int_0^{a+b} k \left( \frac{t}{a+b} \right) \tilde{f}(t) dt \geq k(1)F(a + b) \]

\[ + \int_0^1 k'(t) \{ F(at) + F(a + bt) - F(a) \} dt > (a + b)\alpha, \]

from which the result follows.

4. We are now in a position to prove (3).

From the definition of \( \Theta(x) \) it is seen that to every point \( x \) of \( E \subset \mathbb{R}^1_+ \) there exists at least one interval \( (x - h_x, x) \subset (0, 1) \) for which

\[ \frac{1}{h_x} \int_0^{h_x} k \left( \frac{t}{h_x} \right) f(x - h_x + t) dt > \alpha. \]

Let us apply Lemma 1 to the set \( E \). Then there exists a finite set \( S \) of the above intervals which are nonoverlapping, which cover a subset \( E' \) of \( E \) such that \( mE' > mE - \epsilon \). It is evidently sufficient to prove (3) when we replace \( mE \) in the left-hand side by \( mE' \) for all \( \epsilon > 0 \), and, a fortiori, it is sufficient to prove that

\[ mS \leq mE \left[ x, \frac{1}{x} \int_0^x k \left( \frac{t}{x} \right) \tilde{f}(t) dt > \alpha \right]. \]

We further use the legitimate simplification that \( f(t) = 0 \) for \( t \) in the complement of \( S \), since this decreases the right-hand side of (6). This latter assumption permits us to translate the intervals of \( S \) to the left until they are end to end, and completely cover the interval \( (0, mS) \), i.e., we have the \( n \) intervals \( (j_r, j_{r+1}) \) with \( 0 = j_0 < j_1 < \cdots < j_n = mS \).

Consider the two intervals \( (j_{n-2}, j_{n-1}), (j_{n-1}, j_n) \).

We have

\[ \frac{1}{j_n - j_{n-1}} \int_0^{j_n - j_{n-1}} k \left( \frac{t}{j_n - j_{n-1}} \right) f(t) dt > \alpha, \]

\[ \frac{1}{j_{n-1} - j_{n-2}} \int_0^{j_{n-1} - j_{n-2}} k \left( \frac{t}{j_{n-1} - j_{n-2}} \right) f(t) dt > \alpha. \]
Let \( f_1(t) \) be the decreasing rearrangement of \( f(t) \) within the above two intervals. Then, applying Lemma 2, we have

\[
\frac{1}{j_n - j_{n-2}} \int_0^{j_n - j_{n-2}} k\left(\frac{t}{j_n - j_{n-2}}\right) f_1(t) dt > \alpha.
\]

We have thus reduced the case from \( n \) intervals to \((n-1)\) intervals. Repeating the process a further \((n-2)\) times we arrive at \( \bar{f}(t) \) defined in the single interval \((0, mS)\), and have

\[
\frac{1}{mS} \int_0^{mS} k\left(\frac{t}{mS}\right) \bar{f}(t) dt > \alpha
\]

which proves (6).

Thus we have proved the theorem for the special case (2).

5. Since \( s(t) \) is an increasing function of \( t \), we may approximate to it uniformly in any finite interval \((0, N)\) by the sum of a finite number of functions of the type \( s_a(t) \). The deduction of Theorem 2 from the special case considered above is apparent.

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**Bibliography**


Princeton University