TRANSFORMATIONS OF FOURIER COEFFICIENTS

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Introduction. In 1923 M. Fekete [2][2] introduced the concept of factor sequences that left invariant the class of a Fourier series. That is, Fekete investigated the conditions to which a sequence of constants (λ_n) must be subjected in order that $(\lambda_n a_n, \lambda_n b_n)$ be Fourier coefficients of a function of the same class, K, as that of the function determined by (a_n, b_n). Whenever (λ_n) has this property, (λ_n) is said to belong to the class (K, K). Fekete restricted his investigation to those cases for which K represented the class of continuous, essentially bounded, Riemann integrable, or Lebesgue integrable functions, functions of bounded variation, or functions having a Lebesgue-Stieltjes series. It may be mentioned in passing that Verblunsky [5] extended Fekete's results to some of the cross classes.

Note that the transformation effected by a factor sequence may be accomplished by matric multiplication. Suppose that the cosine coefficients of f are (a_n). Then the coefficients of the function resulting from the transformation appear in the column vector on the right below.

$$\begin{bmatrix} \lambda_0 & 0 & 0 & \cdots \\ 0 & \lambda_1 & 0 & \cdots \\ 0 & 0 & \lambda_2 & \cdots \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} \lambda_0 a_0 \\ \lambda_1 a_1 \\ \lambda_2 a_2 \\ \cdot \\ \cdot \end{bmatrix}.$$

The purpose of this paper is to extend the diagonal matrix on the left to a full $\infty \times \infty$ matrix with constant elements. We shall investigate such matrices, denoted by T, and endeavor to find necessary and sufficient conditions for T to transform the coefficients of a function in the class L_p into coefficients of another function in the same class.

G. H. Hardy [3] has shown that the arithmetic means of the partial sums of coefficients of a function in L_p, $p \geq 1$, are also in the same class. Since this transformation can be accomplished by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \cdots \\ 1/2 & 1/2 & 0 & 0 \cdots \\ 1/3 & 1/3 & 1/3 & 0 \cdots \\ \cdot & \cdot & \cdot & \cdot \cdot \cdot \cdot \cdot \cdot \end{bmatrix},$$

Hardy's paper gives a preliminary result along the present line of investigation. In 1944, Bellman [1] showed that the transpose of the matrix above is also of the same type for $p > 1$. In view of Hardy's theorem, Bellman's result is an immediate consequence of Theorem 5 of this paper.

The following treatment will be seen to be complete for only the open interval, $1 < p < \infty$. The end points will need a more searching analysis.

Definitions and notation. The letter T will be used to denote both the matrix (a_{kj}) and the associated transformation. Further, let

(1) $$(a_j, b_j) \sim f \in L_p, \qquad\qquad 1 < p < \infty.$$

The symbol $\sim$ will be used to indicate the correspondence between an integrable function and its Fourier coefficients. Thus, in (2) and (3) below, the indicated correspondence must be justified. With p as in (1),

(2) for each k,

$$(a_{kj}, 0) \sim g_k \in L_{p'} \quad \text{and} \quad G_{n,k}(x) = \sum_{j=0}^{n} \left(\frac{n+1-j}{n+1} \right) a_{kj} \cos jx,$$

$$(0, a_{kj}) \sim h_k \in L_{p'} \quad \text{and} \quad H_{n,k}(x) = \sum_{j=0}^{n} \left(\frac{n+1-j}{n+1} \right) a_{kj} \sin jx;$$

(3) for each j,

$$(a_{kj}, 0) \sim p_j \in L_p \quad \text{and} \quad P_{n,j}(x) = \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) a_{kj} \cos kx,$$

$$(0, a_{kj}) \sim q_j \in L_p \quad \text{and} \quad Q_{n,j}(x) = \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) a_{kj} \sin kx;$$

(4) $$\overset{T}{\sigma_n}(f; x) = \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) \sum_{j=0}^{\infty} a_{kj}(a_j \cos kx + b_j \sin kx);$$

(5) $$U_n(x; t) = \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) [g_k(t) \cos kx + h_k(t) \sin kx].$$

There is no loss of generality in assuming that the first row of T

consists entirely of zeros. Since we shall also discuss T^*, the transpose of T, we shall also let the first column of T consist of zeros. This enables us to use the simpler expression employed in (2) and (3) and, later, for the Fejér means.

The association of function and series in (2) and (3) is so far purely formal. However, in general,

$$(6) \qquad T(f) \sim \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} a_{kj} a_j \cos kx + \sum_{j=0}^{\infty} a_{kj} b_j \sin kx \right],$$

and for these coefficients to exist for every f in some L_p, $1 < p < \infty$, it is necessary that each row be in $L_{p'}$, which justifies (2). Further, if f is specialized to be either $\cos jx$ or $\sin jx$, it is clear that each column must be in L_p, which justifies (3). We shall say that $T \in (L_p)$ if (1) implies that $T(f) \in L_p$.

LEMMA 1. *For* $1 < p < \infty$, $\sigma_n^T(f; x) = (1/\pi) \int_0^{2\pi} f(t) L_n(x; t) dt$, *where* $L_n(x; t) \sim (P_{n,j}(x), Q_{n,j}(x))$ *for each* n *and* x.

PROOF. By (1) and (2), $\sigma_n^T(f; x)$ exists for each n and all x, $0 \le x \le 2\pi$, $1 < p < \infty$, and

$$\sigma_n^T(f; x) = \sum_{j=0}^{\infty} \left[a_j \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) a_{kj} \cos kx \right.$$

$$\left. + b_j \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) a_{kj} \sin kx \right]$$

$$= \sum_{j=0}^{\infty} [a_j P_{n,j}(x) + b_j Q_{n,j}(x)].$$

Thus, for each n and x, by Parseval's relation,

$$(P_{n,j}(x), Q_{n,j}(x)) \sim L_n(x; t) \in L_{p'},$$

and, moreover, $\sigma_n^T(f; x) = (1/\pi) \int_0^{2\pi} f(t) L_n(x; t) dt$.

LEMMA 2. $\|L_n(x; t)\|_{p'} \le K(n)$ *for all* x, *the norm being taken with respect to* t.

PROOF.

$$L_n(x; t) \sim \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) \left[\left(\sum_{j=0}^{\infty} a_{kj} \cos jt \right) \cos kx \right.$$

$$\left. + \left(\sum_{j=0}^{\infty} a_{kj} \sin jt \right) \sin kx \right].$$

Let $\sigma_m(L_n(x; t); t)$ be the mth mean of $L_n(x; t)$ with respect to t. Then

$$\sigma_m(L_n(x; t); t) = \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1}\right) [G_{m,k}(t) \cos kx + H_{m,k}(t) \sin kx].$$

By Minkowski's inequality, again with respect to t,

$$\left\|\sigma_m(L_n(x; t); t)\right\|_{p'} \leqq \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1}\right) [\|G_{m,k}\|_{p'} + \|H_{m,k}\|_{p'}]$$

$$\leqq \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1}\right) [\|g_k\|_{p'} + \|h_k\|_{p'}],$$

which is independent of both x and m. Hence the lemma follows.

Define $\|L_n(x; t)\|_{p'} = M_n(x)$. Then $M_n(x) \in B$ for each n.

We now have

$$L_n(x; t) \sim \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1}\right) \sum_{j=0}^{\infty} a_{kj} \cos (kx - jt).$$

Recall that

$$U_n(x; t) = \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1}\right) (g_k(t) \cos kx + h_k(t) \sin kx).$$

Then the Fourier series of $U_n(x; t)$ is the same as that of $L_n(x; t)$. That is, for each x, $U_n(x; t) = L_n(x; t)$ for almost all t. Thus, we may write

$$\sigma_n^T(f; x) = \frac{1}{\pi} \int_0^{2\pi} f(t) U_n(x; t) dt$$

$$= \frac{1}{\pi} \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1}\right) \int_0^{2\pi} f(t) [(g_k(t) \cos kx + h_k(t) \sin kx] dt.$$

We note that for each x, $U_n(x; t)$ is in $L_{p'}$ as a function of t, and for each t, $U_n(x; t)$ is a continuous function of x.

THEOREM 1. *If* $T \in (L_p)$, *then* T *is a linear bounded transformation.*

PROOF. $|\sigma_n^T(f; x)| \leqq (1/\pi)\|f\|_p M_n(x)$ where $M_n(x) \in B$. $\|T_n(f)\|_p = \|\sigma_n^T(f; x)\|_p \leqq (1/\pi)\|f\|_p K(n)$. Then T_n is a bounded linear transformation for each n with modulus $\leqq K(n)/\pi$. However, $\|T_n(f)\|_p$ is bounded for each $f \in L_p$. Hence, the sequence of moduli is bounded by the principle of uniform boundedness. Moreover, $T_n(f) \to T(f)$ in the metric of L_p. Thus, by the Banach-Steinhaus theorem, T is a bounded linear transformation.

The result we shall use is that $\left\| T_n(f) \right\|_p \leq K \|f\|_p$ for all n.

It is known that $\left\| \sigma_n^T(f;\, x) \right\|_p = \sup_{\|g\|_{p'} \leq 1} \left| \int_0^{2\pi} g(x) \sigma_n^T(f;\, x) dx \right|$. Fix n and f. Then a particular g with $\|g\|_{p'} \leq 1$ is determined such that for that n, f, and g,

$$\left\| \sigma_n^T(f;\, x) \right\|_p = \left| \int_0^{2\pi} g(x) \sigma_n^T(f;\, x) dx \right|$$

$$(7) \qquad = \frac{1}{\pi} \left| \int_0^{2\pi} g(x) \int_0^{2\pi} f(t) U_n(x;\, t) dt dx \right|$$

$$= \frac{1}{\pi} \left| \int_0^{2\pi} f(t) \left\{ \int_0^{2\pi} g(x) U_n(x;\, t) dx \right\} dt \right|.$$

Theorem 2. *A necessary and sufficient condition for $T \in (L_p)$, $1 < p < \infty$, is the existence of K such that $\sup_{\|g\|_{p'} \leq 1} \left\| \int_0^{2\pi} g(x) U_n(x;\, t) dx \right\|_{p'} \leq K$ for all n.*

Proof. *Sufficiency.* Let $f \in L_p$ and fix n. Then let g be the function for which (7) holds. Then, by Hölder's inequality, $\left\| \sigma_n^T(f;\, x) \right\|_p \leq (1/\pi) \|f\|_p \left\| \int_0^{2\pi} g(x) U_n(x;\, t) dx \right\|_{p'} \leq (K/\pi) \|f\|_p$. Thus, $T \in (L_p)$.

Necessity. Let $T \in (L_p)$. Fix n and g with $\|g\|_{p'} \leq 1$ and let $T_n(f) = (1/\pi) \int_0^{2\pi} f(t) \left\{ \int_0^{2\pi} g(x) U_n(x;\, t) dx \right\} dt$. Then $\left| T_n(f) \right| \leq \left\| \sigma_g^T(f;\, x) \right\|_p \leq K \|f\|_p$ by Theorem 1. Thus, $T_n(f)$ is a linear functional over L_p, and, by Banach, its norm is $\left| T_n \right| = \left\| \int_0^{2\pi} g(x) U_n(x;\, t) dx \right\|_{p'} \leq K$ for each n and g.

It is now possible to indicate other conditions that are either necessary or sufficient for $T \in (L_p)$.

Theorem 3. *A sufficient condition for $T \in (L_p)$ is given by either*

$$(1) \qquad \left\| \left(\left\| U_n(x;\, t) \right\|_{p'} \right) \right\|_p = O(1),$$
$$ {}_{t} {}_{x}$$

or

$$(2) \qquad \left\| \left(\left\| U_n(x;\, t) \right\|_p \right) \right\|_{p'} = O(1).$$
$$ {}_{x} {}_{t}$$

Proof. Part (1) follows immediately from an application of Hölder's inequality to the right side of

$$\left| \sigma_n^T(f;\, x) \right| = \left| \frac{1}{\pi} \int_0^{2\pi} f(t) U_n(x;\, t) dt \right|.$$

Part (2) is a corollary to Theorem 2.

Theorem 4. $T \in (L_p)$ *implies* $\|g_n\|_{p'}$ *and* $\|h_n\|_{p'}$ *are uniformly bounded.*

Proof. If $T \in (L_p)$, then $\sup_{\|g\|_{p'} \leq 1} \left\| \int_0^{2\pi} g(x) U_n(x;\, t) dx \right\|_{p'} \leq K$. Also,

$$U_n(x; t) = \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) (g_k(t) \cos kx + h_k(t) \sin kx).$$

Let $g(x) = \cos jx$, $j \leqq n$. Then

$$\int_0^{2\pi} g(x) U_n(x; t) dx = \pi g_j(t) \left(\frac{n+1-j}{n+1} \right) \qquad \text{for all } n \geqq j.$$

Thus,

$$\pi \left(\frac{n+1-j}{n+1} \right) \|g_j(t)\|_{p'} \leqq K, \quad \text{or} \quad \|g_j(t)\|_{p'} \leqq \frac{K}{\pi} \left(\frac{n+1}{n+1-j} \right)$$
$$\text{for all } n \geqq j.$$

For each j, we can choose n so large that $((n+1)/(n+1-j)) \leqq 2$. Hence, $\|g_j(t)\|_{p'} \leqq 2K/\pi$. Similarly, $\|h_k(t)\|_{p'} \leqq 2K/\pi$.

THEOREM 5. *Let T^* be the transpose of T. If $T \in (L_p)$, $1 < p < \infty$, then $T^* \in (L_{p'})$.*

PROOF. Let $g \sim (c_k, d_k)$, $g \in L_{p'}$. Then

$$T^*(g) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{kj}(c_k \cos jt + d_k \sin jt).$$

Consider

$$F_n(g; t) = \frac{1}{\pi} \int_0^{2\pi} g(x) U_n(x; t) dx$$

$$\sim \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) \sum_{j=0}^{\infty} a_{kj}(c_k \cos jt + d_k \sin jt).$$

Since $T \in (L_p)$, $\|F_n(g; t)\|_{p'} \leqq K$ for all n, by Theorem 2. Let

$$\sigma_m(F_n) = \sum_{j=0}^{m} \left(\frac{m+1-j}{m+1} \right) \sum_{k=0}^{n} \left(\frac{n+1-k}{n+1} \right) a_{kj}(c_k \cos jt + d_k \sin jt).$$

Then $\|\sigma_m(F_n)\|_{p'} \leqq K$ for all n. By Fatou's lemma,

$$\int_0^{2\pi} \lim_{n \to \infty} |\sigma_m(F_n)|^{p'} dt \leqq \liminf_{n \to \infty} \int_0^{2\pi} |\sigma_m(F_n)|^{p'} dt,$$

or $\|\sigma_m(T^*g)\|_{p'} \leqq K$, which is sufficient for $T^* \in (L_{p'})$.

COROLLARY. *If $T \in (L_p)$, $1 < p < \infty$, then there is a uniform bound to the p'th norms of the rows and the pth norms of the columns.*

PROOF. This follows from Theorem 4.

THEOREM 6. *Let $T\in(L_p)$ and (L_q). Then $T\in(L_r)$ for any r between p and q.*

PROOF. By Theorem 1, T may be considered to be a linear bounded transformation from L_p to L_p and L_q to L_q. Then, by the well known interpolation theorem of M. Riesz [7, p. 198], $T\in(L_r)$ for any r between p and q.

THEOREM 7. *Every bounded linear transformation T on L_p to L_p can be represented uniquely by a matrix transformation.*

PROOF. In complex form, let $f\sim(c_k)\in L_p$. Let $s_m=\sum_{j=-m}^{m}c_je^{ijx}$. Since $T(f)\in L_p$, let $T(f)=F\sim(d_k)$. Since $e^{ijx}\in L_p$ for each j, consider the Fourier expansion, $T(e^{ijx})\sim\sum_{k=-\infty}^{\infty}a_{kj}e^{ikx}$, where $a_{kj}=(1/2\pi)\int_0^{2\pi}[T(e^{ijx})]e^{-ikx}dx$. Then $T(s_m)\sim\sum_{j=-m}^{m}c_j\sum_{k=-\infty}^{\infty}a_{kj}e^{ikx}$, and if $c_k[T(s_m)]$ is the kth coefficient in the Fourier expansion of $T(s_m)$, then

$$c_k[T(s_m)] = \sum_{j=-m}^{m}a_{kj}c_j.$$

Since $\|s_m-f\|_p=o(1)$, the continuity of T implies that $\|T(s_m)-F\|_p=o(1)$, from which it follows that $\lim_{m\to\infty}c_k[T(s_m)]=d_k$. That is, $d_k=\sum_{j=-\infty}^{\infty}a_{kj}c_j$.

The uniqueness of the representation follows from the uniqueness of the Fourier series expansion of a function in L_p.

It is also possible to give necessary and sufficient conditions for $T\in(L_p)$ in another form. The following is superficially similar to a theorem by Hille and Tamarkin [4] concerning (L_p)-effective matrices.

THEOREM 8. *Necessary and sufficient conditions for T to be in (L_p), $1<p<\infty$, are:*

(i) *$T\in(L_p)$ on every subset L_p^0 which is dense in L_p, and*

(ii) *the set of transformations $\{T_n\}$ defined by $T_n(f)=\sigma_n^T(f;x)$ form a uniformly bounded sequence of transformations on L_p into itself. For sufficiency, condition (i) may be replaced by*

(i′) *$T\in(L_p)$ on a single subset L_p^0 which is dense in L_p.*

PROOF. *Necessity.* The necessity of (i) is obvious. The necessity of (ii) is an immediate consequence of the proof of Theorem 1.

Sufficiency. Assume that (i′) and (ii) are satisfied. Then an arbitrary function f in L_p can be approximated in L_p as closely as we wish by a function of L_p^0. That is, if we are given an f in L_p and an arbitrarily

small $\epsilon > 0$, then we can find a function g_ϵ in L_p^0 such that $\|f - g_\epsilon\|_p \leq \epsilon/3K$, where K is an upper bound to the moduli of the operations defined by $\sigma_n^T(f; x)$. Since g_ϵ is in L_p, there exists an $N > 0$ such that for $n, m \geq N$, $\|\sigma_n^T(g_\epsilon; x) - \sigma_m^T(g_\epsilon; x)\|_p \leq \epsilon/3$. We may write $\|\sigma_n^T(f; x) - \sigma_m^T(f; x)\|_p \leq \|\sigma_n^T(f - g_\epsilon)\|_p + \|\sigma_n^T(g_\epsilon) - \sigma_m^T(g_\epsilon)\|_p + \|\sigma_m^T(g_\epsilon - f)\|_p$. Then, for $n, m \geq N$, $\|\sigma_n^T(f; x) - \sigma_m^T(f; x)\|_p \leq K\epsilon/3K + \epsilon/3 + K\epsilon/3K = \epsilon$. Since the space L_p is complete, there exists an F in L_p such that $\|\sigma_n^T(f; x) - F(x)\|_p$ converges to zero. This permits us to say that $T(f) = F$ in L_p, or $T \in (L_p)$.

So far, the end cases, $T \in (L)$ and $T \in (B)$, have defied our efforts. However, if we retreat to a more restricted class of matrices, those for which $\sum_{j=0}^{\infty} |a_{kj}| < M$ for all k, certain further results may be obtained. First of all, the integral representation of Lemma 1 is valid for the entire range, $1 \leq p \leq \infty$. With this representation it is not difficult to demonstrate the following theorem [6].

THEOREM 9. *If* $\sum_{j=0}^{\infty} |a_{kj}| < M$ *for all* k, *a necessary and sufficient condition for* $T \in (B)$ *is that* $\int_0^{2\pi} |L_n(x; t)| \, dt$ *be bounded uniformly in* n *and* x.

For the other end point, we may summarize the partial results obtained in the following [6]:

THEOREM 10. *If* $\sum_{j=0}^{\infty} |a_{kj}| < M$ *for all* k, *a necessary condition for* $T \in (L)$ *is* $\int_0^{2\pi} |L_n(x; t)| \, dx \leq K$ *for all* n *and* t. *Further, sufficient conditions for* $T \in (L)$ *are given by either of the following:*
    (i) $\int_0^{2\pi} |L_m(x; t) - L_n(x; t)| \, dx < \epsilon$ *for large* m, n, *independent of* t;
    (ii) (a) $\int_0^{2\pi} |L_n(x; t)| \, dt \leq K_1$ *for all* n *and* x, *and*
    (b) $\int_0^{2\pi} |L_n(x; t)| \, dx \leq K_2$ *for all* n *and* t.

It may also be observed that if T and T^* are subjected to the condition of row boundedness used above, then $T^* \in (L)$ implies $T \in (B)$ [6].

The proofs of Theorems 9 and 10 and of the last remark will be omitted since the results are so markedly weaker than those for the open interval.

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A REMARK ON LINEAR ELLIPTIC DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Consider a linear partial differential expression

$$L(u) = \sum_{i,k} a_{ik}(x) \frac{\partial u}{\partial x_i \partial x_k} + \sum_i b_i(x) \frac{\partial u}{\partial x_i}$$

with no term $c(x)u$. The coefficients a_{ik} and b_i are suppose to be continuous in an open connected set R of x-space, $x = (x_1, \cdots, x_n)$. Let x^0 denote a point on the boundary of R which has the property that R contains the interior of a hypersphere $|x - x^*| < r_0$ with x^0 on its boundary. Suppose that the coefficients are continuous at $x = x^0$ also. Let, finally, L be elliptic in $R + x^0$ such that the quadratic form

$$\sum a_{ik}(x)\lambda_i\lambda_k$$

is positive definite in each point of $R + x^0$.

This note contains a simple proof of the following:

THEOREM. *Suppose that $u = u(x)$ is of class C'' in R and that $u \geq 0$, $L(u) \leq 0$ in R. If the limit value of u at $x = x^0$ is zero, then either the normal derivative du/dn at $x = x^0$, understood as the limit inferior of $\Delta u/\Delta n$, is > 0 or $u \equiv 0$ in R.*

Special cases of the theorem have been known for a long time. It contains, in particular, the fact that Green's function of L has a positive normal derivative along the boundary if the boundary is sufficiently smooth.

To prove the theorem we note first that $u \geq 0$ in R and $u(x^0) = 0$ trivially implies $du/dn \geq 0$. The hypotheses that $u \geq 0$ in R and $L(u) \leq 0$ in R imply that either $u > 0$ in R or $u \equiv 0$ in R. This follows from