A THEOREM ABOUT FRACTIONAL INTEGRALS

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It is a classical result that if $f(x)$ is Lebesgue integrable in a finite interval, then it is finite p.p. One is led to enquire about the behaviour of the fractional integral $f_\beta$ of $f$.

Suppose, for convenience, that $f(x)$ is defined in $[0, 2\pi]$. We then have the following theorem.

**Theorem.** If $f \in L^q[0, 2\pi]$, then:

(a) For $0 < \alpha < 1$, $2 < q < \infty$, $f_{\alpha/q}$ is finite everywhere except in a set which is of zero $\beta$-capacity for every $\beta > 1 - \alpha$.

(b) For $0 < \alpha < 1$, $1 \leq q \leq 2$, $f_{\alpha/q}$ is finite everywhere except possibly in a set of zero $(1 - \alpha)$-capacity.

Both (a) and (b) are best possible.

Since, as is well known, the Riemann-Liouville and the Weyl versions of the fractional integral of a function in $L^q$ differ by a bounded function, this theorem holds for both versions if it is shown to hold for either one. Use is made of this fact in what follows.

1. In this section I prove a lemma which is possibly of greater interest than the theorem itself.

**Lemma.** Let $\mu(x)$ be a nondecreasing bounded function in $[0, 2\pi]$. Let, for $0 < \alpha < 1$,

$$V_{1-\alpha} = \sup \int_0^{2\pi} |x - t|^\alpha \mu(t) \text{ for } x \in [0, 2\pi].$$

Then, for every $\epsilon > 0$ and $1 < q < 2$,

$$M_{q-\epsilon} \left[ \int_0^{2\pi} |x - t|^{\alpha/q' - 1} \mu(t) \right] \leq A(\alpha, \epsilon) V_{1-\alpha},$$

where $A(\alpha, \epsilon)$ is a constant depending only on $\alpha$ and $\epsilon$, and, for $2 \leq q \leq \infty$,

$$M_q \left[ \int_0^{2\pi} |x - t|^{\alpha/q - 1} \mu(t) \right] \leq A(\alpha) V_{1-\alpha},$$

where $A(\alpha)$ is a constant depending only on $\alpha$.

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\[ \int_0^{2\pi} |x - t|^{\alpha/q' - 1} d\mu(t) = \int_0^{2\pi} |x - t|^{-a/q} dv_x(t), \]

where

\[ v_x(t) = \int_0^t |x - s|^{\alpha - 1} d\mu(s). \]

Consequently, using Hölder's inequality,

\[ \left\{ \int_0^{2\pi} |x - t|^{\alpha/q' - 1} d\mu(t) \right\}^{q - t} \leq \left\{ \int_0^{2\pi} |x - t|^{-a(q-1)/q} dv_x(t) \right\} \left\{ \int_0^{2\pi} dv_x(t) \right\}^{(q-1)/(q-1)'} \]

and this does not exceed

\[ \left\{ \int_0^{2\pi} |x - t|^{a/q - 1} d\mu(t) \right\} V^{(q-1)/(q-1)'}_{\beta - a}. \]

Thus, the left-hand side of (1) does not exceed

\[ V^{1/(q-1)'} \left\{ \int_0^{2\pi} d\mu(t) \int_0^{2\pi} |x - t|^{a/q - 1} dx \right\}^{1/(q-1)}, \]

which gives (1). It is surprising that so crude an argument gives a best possible result.

To prove (2) I first show the result true for \( q = 2 \) and then that this implies its truth for \( q > 2 \). For this latter portion of the proof I am indebted to Professor J. E. Littlewood.

We have first, inverting the order of integration,

\[ M^2 \left[ \int_0^{2\pi} |x - t|^{a/2 - 1} d\mu(t) \right] \]

(3)

\[ = \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |x - t|^{a/2 - 1} |x - s|^{a/2 - 1} d\mu(t) d\mu(s). \]

In the inner integral we make the substitution \( x - t = (s - t)u \) and find that the integral does not exceed

\[ |s - t|^{a-1} \int_{-\infty}^{\infty} |u(1 - u)|^{a/2 - 1} du = B(\alpha) |s - t|^{a-1}. \]

Consequently, the right-hand side of (3) is dominated by
which gives the result for \( q = 2 \).

For \( q > 2 \), we have

\[
M_q^2 \left[ \int_0^{2\pi} \left| x - t \right|^{a/q - 1} d\mu(t) \right]^{\frac{q}{2}} = \int_0^{2\pi} dx \left\{ \int_0^{2\pi} \left| x - t \right|^{(a-2)/(q-1)} | x - t |^{(a-q)/q} d\mu(t) \right\}^q
\]

and this, by Hölder's inequality, does not exceed

\[
\int_0^{2\pi} dx \left\{ \int_0^{2\pi} \left| x - t \right|^{a - 1} d\mu(t) \right\} \int_0^{2\pi} dx \left\{ \int_0^{2\pi} \left| x - t \right|^{a - 1} d\mu(t) \right\}^q,
\]

which is, in turn, dominated by

\[
V_{1-a}^q M_q^2 \left[ \int_0^{2\pi} \left| x - t \right|^{a/2 - 1} d\mu(t) \right] \leq A(\alpha) V_{1-a}^{q-1},
\]

which gives the result for \( q > 2 \).

2. Proof of the theorem. Suppose that \( E \) is a subset of \([0, 2\pi]\).

If a nondecreasing \( \mu(x) \) is such that

\[
\int_E d\mu(t) = \int_0^{2\pi} d\mu(t) = 1,
\]

we say that \( \mu(x) \) is a distribution concentrated on \( E \). If, further, for any \( \beta \) \((0 < \beta < 1)\) there is a \( \mu(x) \) concentrated on \( E \) such that

\[
V_{\beta} = \sup \int_0^{2\pi} \left| x - t \right|^{-\beta} d\mu(t) \quad \text{for all } x \in [0, 2\pi]
\]

is finite, then \( E \) is said to be of positive \( \beta \)-capacity. Otherwise \( E \) is said to be of zero \( \beta \)-capacity. This definition is equivalent to that given by Salem and Zygmund \([1]\).

Clearly, if \( E \) is of positive \( \beta \)-capacity, it is of positive \( \gamma \)-capacity for all \( \gamma < \beta \). If it is of zero \( \beta \)-capacity, it is of zero \( \gamma \)-capacity for all \( \gamma > \beta \).

We may, without loss of generality, assume \( \int_0^{2\pi} f(x) dx = 0 \) and use the Weyl fractional integral.

Let \( f(x) \) have the Fourier series

\[
B(\alpha) \int_0^{2\pi} \int_0^{2\pi} | s - t |^{a-1} d\mu(t) d\mu(s) \leq B(\alpha) V_{1-a}(\mu(2\pi) - \mu(0)),
\]
where \( t \) signifies that \( c_0 = 0 \). Then

\[
\sum_{n=-\infty}^{\infty} c_n e^{inz}
\]

and it is sufficient to show that \( S_n \) is bounded outside a set of zero \( \beta \)-capacity, where \( S_n = \sum_{k=-n}^{n} (ik)^{-a/q} c_k e^{ikz} \) and \( \beta = 1 - \alpha \) for \( 1 \leq q \leq 2 \) and \( \beta > 1 - \alpha \) for \( q > 2 \).

Assume then that \( S_n \) is unbounded in a set \( E \) of positive \( \beta \)-capacity. Then, first, there is a distribution \( \mu(x) \) concentrated on \( E \) such that

\[
\int_{0}^{2\pi} \left| x - t \right|^{-\beta} dt \mu(t)
\]

is bounded for all \( x \). Secondly, by a well known argument there is a function \( n(x) \leq n \), taking only integer values, such that

\[
\int_{0}^{2\pi} S_n(x) d\mu(x)
\]

exists and is unbounded as \( n \to \infty \). I show this last to be impossible.

For

\[
\int_{0}^{2\pi} S_n(x) d\mu(x) = \int_{0}^{2\pi} \sum_{k=-n}^{n} (ik)^{-a/q} c_k e^{ikz} d\mu(x)
\]

\[
= \int_{0}^{2\pi} \int_{0}^{2\pi} f(t) \sum_{k=-n}^{n} (ik)^{-a/q} e^{inz} dt d\mu(x).
\]

Now

\[
\left| \sum_{k=-n}^{n} (ik)^{-a/q} e^{inz} \right| \leq C \left| x - t \right|^{-\alpha q - 1}
\]

so

\[
\left| \int_{0}^{2\pi} S_n(x) d\mu(x) \right| \leq C \int_{0}^{2\pi} \left| f(t) \left\{ \int_{0}^{2\pi} \left| x - t \right|^{-\alpha q - 1} d\mu(x) \right\} dt \right.
\]

\[
\leq CM_{q}(f) M_{q'} \left[ \int_{0}^{2\pi} \left| x - t \right|^{-\alpha q - 1} d\mu(x) \right].
\]

Now \( M_q(f) < \infty \) by hypothesis and we have only to show
bounded.

If \( 1 \leq q \leq 2 \), then \( q' \geq 2 \) and (2) of §1 immediately shows this.

If \( q > 2 \), we write \( \beta = 1 - \gamma \). Since \( \gamma < \alpha \), there is an \( r < q \) such that \( \alpha/q = \gamma/r \). We may suppose \( \beta \) so near to \( 1 - \alpha \) that \( 2 < r < q \), since if we show the result for all such \( \beta \) it will immediately follow for all larger \( \beta \). We may now rewrite (1) in the form

\[
M_{q'} \left[ \int_0^{2\pi} |x - t|^{\alpha/q - 1} d\mu(x) \right]
\]

which, since \( r' < 2 \), is shown to be bounded by invoking (1) of §1.

This gives the result.

3. The theorem is best possible. Construct the set \( S \) as follows.

Let \( \{\xi_n\} \) be any sequence such that \( 0 < \xi_n < 1/2 \). From \( S_0 = [0, 2\pi] \) remove the open concentric interval of length \( 2\pi(1 - 2\xi_1) \), thus leaving the set \( S_1 \). From each of the intervals in \( S_1 \), of length \( 2\pi \xi_1 \), remove an open concentric interval of length \( 2\pi \xi_1(1 - 2\xi_2) \), leaving a set \( S_2 \) consisting of four closed intervals each of length \( 2\pi \xi_1 \xi_2 \). Continuing in this way, we are left, after the \( k \)th removal, with a set \( S_k \) consisting of \( 2^k \) closed intervals each of length \( 2\pi \xi_1 \xi_2 \cdots \xi_k \). Consequently \( m_{S_k} = 2\pi 2^k \xi_1 \xi_2 \cdots \xi_k \).

It is known [1, p. 40] that \( S = \lim S_k \) will be of positive \( \beta \)-capacity if and only if

\[
\sum_{k=1}^{\infty} 2^{-k}(\xi_1 \xi_2 \cdots \xi_k)^{-\beta} < \infty.
\]

Define \( \{f_n(x)\} \) on \( [0, 2\pi] \) by

\[
\begin{align*}
f_0(x) &= 0 \quad \text{in} \quad S_0, \\
f_n(x) &= (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha/q - 1} \quad \text{in} \quad S_n \\
&= f_{n-1}(x) \quad \text{in} \quad S_0 - S_n
\end{align*}
\]

for \( n = 1, 2, 3 \cdots \). Since \( \{f_n(x)\} \) is an increasing sequence of measurable functions,

\[
f(x) = \lim_{n \to \infty} f_n(x)
\]

exists and is measurable.

It is easily seen that
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\[ f(x) = (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha/q - 1} \text{ on } S_n - S_{n+1}, \quad n = 1, 2, \ldots, \]

so that

\[
\int_0^{2\pi} |f(x)|^q \, dx = \sum_{n=1}^\infty \int_{S_n - S_{n+1}} |f(x)|^q \, dx
\]

\[
= \sum_{n=1}^\infty (\xi_1 \xi_2 \cdots \xi_n)^{-\alpha n - q} [mS_n - mS_{n+1}]
\]

\[
= \sum_{n=1}^\infty (1 - 2\xi_{n+1})2^n(\xi_1 \xi_2 \cdots \xi_n)^{1-\alpha n - q}.
\]

For \( q > 2 \) we may choose \( \delta > 0 \) so that \( 2(1+\delta) < q \) and then put

\[ 2\xi_{n+1}^{1-\alpha} = 1 + (1 + \delta)n^{-1}. \]

Then

\[ 2^{-k}(\xi_1 \xi_2 \cdots \xi_k)^{\alpha - 1} = O(k^{-1-\delta}) \]

and so (1) with \( \beta = 1 - \alpha \) is satisfied, showing \( S \) to be of positive \((1-\alpha)\) capacity.

Further (2) is clearly finite so that \( f(x) \in L^q \).

Let \( x \) be any point of \( S \). Now \( S_k - S_{k+1} \) consists of \( 2^k \) intervals each of length \( 2\pi \xi_1 \xi_2 \cdots \xi_k (1 - 2\xi_{k+1}) \) none of which contain \( x \). However, one of these intervals \( I_k \) is contained in an interval of \( S_k \) which contains \( x \). Consequently

\[
\int_0^{2\pi} |x - t|^{\alpha/q - 1} f(t) \, dt = \sum_{k=1}^\infty \int_{S_k - S_{k+1}} \sum_{k=1}^\infty \int_{I_k}
\]

and this last is itself greater than

\[
(2\pi)^{\alpha/q} \sum_{k=1}^\infty (\xi_1 \xi_2 \cdots \xi_k)^{\alpha/q - 1} (\xi_1 \xi_2 \cdots \xi_k)^{-\alpha/q} (1 - 2\xi_{k+1})
\]

\[
= (2\pi)^{\alpha/q} \sum_{k=1}^\infty (1 - 2\xi_{k+1}) k^{-1} = + \infty
\]

and so \( \int_0^{2\pi} |x - t|^{\alpha/q - 1} f(t) \, dt = + \infty \) at every point of \( S \).

Now \( f(x) = f(2\pi - x) \) so that

\[
\int_0^{2\pi} |x - t|^{\alpha/q - 1} f(t) \, dt = f_{\alpha/q}(x) + f_{\alpha/q}(2\pi - x)
\]

where \( f_{\alpha/q} \) now denotes the Riemann-Liouville fractional integral.

It follows that \( f_{\alpha/q}(x) \) must be infinite in a set of positive \((1-\alpha)\)-
capacity. For, if not, suppose \( f_{a/q}(x) \) infinite in a set \( M \) of zero 
\((1-\alpha)\)-capacity. Then \( f_{a/q}(2\pi-x) \) is infinite in the mirror-image \( M \) of 
\( M \) about \( x=\pi \). Also \( M + M = S \), and since both \( M \) and \( M \) are of 
zero \((1-\alpha)\)-capacity, so is \( S \). This contradiction gives the required 
result and shows part (b) of the theorem to be best possible.

Next, let \( \beta \) be any positive number less than \( 1-\alpha \) and let \( \xi \) be such 
that
\[
2\xi^{1-\alpha+\beta}/2 = 1.
\]
Now consider the set \( S \) with \( \xi_n = \xi \) for all \( n \). Since \( 2\xi^q > 1 \), \( S \) is of posi-
tive \( \beta \)-capacity. Defining \( f(x) \) as before we use exactly the same argu-
ment to show \( f_{a/q}(x) = +\infty \) in a set of positive \( \beta \)-capacity. Further-
more, since \( 2\xi^{1-\alpha} < 1 \), (2) is bounded, showing \( f \in L^q \). This proves 
part (a) of the theorem to be best possible.

In passing, we may note that it has here been shown that a func-
tion in \( L^q \) for any \( q \) may be infinite in a set which is “only just” of 
measure zero. More precisely, given any \( \beta < 1 \) and any \( q > 1 \), there is 
a function in \( L^q \) which is infinite in a set of positive \( \beta \)-capacity, i.e., 
in a set of positive \( \beta \)-Hausdorff measure.

4. The lemma of §1 is best possible. Consider e.g., (2) of §1. Sup-
pose this is not best possible, i.e., that there is an \( \epsilon > 0 \) for which, in 
general,
\[
M_{q+\epsilon}
\left[
\int_0^{2\pi} x - t \left| a/q^\prime - 1 \right| d\mu(t) \right] < \infty.
\]
If, then, \( f(x) \in L^{(q+\epsilon)^\prime} \), we may say
\[
\left| \int_0^{2\pi} S_n(x)(x) d\mu(x) \right| \leq CM_{(q+\epsilon)^\prime}(f) M_{q+\epsilon}
\left[
\int_0^{2\pi} x - t \left| a/q^\prime - 1 \right| d\mu(t) \right],
\]
which is bounded. This would imply that (b) of the theorem is not 
best possible. Since it is, we have shown (2) best possible. A similar 
argument using (a) would show (1) best possible.

References

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