ON THE INTEGRAL EQUATION $\lambda f(x) = \int_0^a K(x-y)f(y)dy$

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1. Introduction. We wish to consider the integral equation

\begin{equation}
\lambda f(x) = \int_0^a K(x-y)f(y)dy, \quad a > 0,
\end{equation}

which occurs in connection with various problems of probability theory and mathematical physics. Unless $K(x)$ is a function of particularly simple type, such as a polynomial or sum of exponentials, the problem of obtaining an exact solution of (1) appears exceedingly difficult. In the present note we discuss the behavior of the largest characteristic value, $\lambda_M$, as $a \to \infty$, under certain assumptions concerning $K(x)$, and illustrate our results with reference to the integral equation of Kac,

\begin{equation}
\lambda f(x) = \int_0^a e^{-(x-y)^2}f(y)dy.
\end{equation}

The principal result is

**Theorem 1.** If

(a) $K(x)$ is non-negative, even, and monotone decreasing for $0 \leq x < \infty$,

(b) $c = \int_0^\infty K(x)dx < \infty$,

then as $a \to \infty$, $\lambda_M \to 2c$.

More precisely, for all $a > 0$,

\begin{equation}
2 \int_0^{a/2} K(x)dx \geq \lambda_M \geq 2 \int_0^a K(x)dx - \frac{2}{a} \int_0^a xK(x)dx.
\end{equation}

Our first method of proof depends upon two tools, the classical Rayleigh-Ritz procedure and a new variational procedure introduced by Bohnenblust. The second method utilizes some known techniques of the theory of integral equations, and exhibits an important property of the characteristic function associated with $\lambda_M$.

2. First proof. We shall employ the following two lemmas, the first of which is well known:

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Lemma 1. If $K(x, y)$ is real, symmetric, and satisfies the condition
that
$$\int_0^a \int_0^a K^2(x, y) dx dy < \infty,$$
then
$$\lambda_M = \text{Max} \frac{\int_0^a \int_0^a K(x, y)f(x)f(y) dx dy}{\int_0^a f^2(x) dx}.$$

Lemma 2. If $K(x, y)$ is bounded and non-negative for $0 \leq x, y \leq a$, and $\lambda_M$ denotes, as above, the largest characteristic value of $K(x, y)$, then
$$\sup \frac{\int_0^a K(y, x)g(y) dy}{\min \frac{g(x)}{\inf \text{Max} \int_0^a K(y, x)g(y) dy}} \leq \lambda_M.$$  

Proof of Lemma 2. As is known, the characteristic function associated with $\lambda_M$ may be taken to be positive, by virtue of the non-negativity of $K(x, y)$, taking $K$ to be nontrivial. Let $g(x)$ be a positive function greater than or equal to one. From
$$\lambda_M f(x) = \int_0^a K(x, y)f(y) dy,$$
we obtain
$$\lambda_M \int_0^a f(x)g(x) dx = \int_0^a \left( \int_0^a K(x, y)g(x) dx \right) f(y) dy$$
$$= \int_0^a \frac{\left( \int_0^a K(x, y)g(x) dx \right)}{g(y)} f(y)g(y) dy$$
whence (2) follows immediately. That the two sides of the inequality in (2) are actually equal and equal to $\lambda_M$ is a result of Bohnenblust.
Lemma 1 contains the essence of the Rayleigh-Ritz method and furnishes lower bounds for $\lambda_M$. Lemma 2, which is also based upon variational principles, furnishes upper and lower bounds. Combining the two, and using the fact that $K(x)$ is even, we obtain

$$\inf_{J} \max_{0 \leq x \leq a} K(x - y)g(y)dy \geq \lambda_M$$

(5)

$$= \max_{f} \int_{0}^{a} \int_{0}^{a} K(x - y)f(x)f(y)dxdy$$

$$\geq \sup_{J} \min_{0 \leq x \leq a} \int_{0}^{a} K(x - y)g(y)dy$$

The simplest possible choices of $f$ and $g$, viz., $f=g=1$, yield (4) of §1. It is clear that these results may be further refined by a cleverer choice of $f$ and $g$. However, the calculations rapidly become complicated.

Setting $f=1$, we obtain

$$\lambda_M \geq \int_{0}^{a} \left[ \int_{0}^{a} K(x - y)dy \right] dx/a$$

(6)

$$= \frac{1}{a} \int_{0}^{a} \left[ \int_{0}^{x} K(u)du + \int_{x}^{a} K(u)du \right] dx$$

$$= \frac{2}{a} \int_{0}^{a} \left[ \int_{0}^{x} K(u)du \right] dx.$$ Integration by parts yields

$$\lambda_M \geq 2 \int_{0}^{a} K(u)du - \frac{2}{a} \int_{0}^{a} uK(u)du.$$

(7)

Setting $g=1$, we obtain

$$\max_{0 \leq x \leq a} \int_{0}^{a} K(x - y)dy \geq \lambda_M.$$

(8)

Since $K$ is even and monotone decreasing, it is easily seen that the maximum occurs at $x=a/2$. Thus,
\begin{equation}
\int_0^a K\left(\frac{a}{2} - y\right) dy = 2 \int_0^{a/2} K(y) dy \geq \lambda_M.
\end{equation}

If \( \int_0^a K(x) dx < \infty \), it follows readily that \( \int_0^a x K(x) dx = o(a) \) as \( a \to \infty \), and thus \( \lambda_M \to 2 \int_0^a K(u) du \) as \( a \to \infty \).

The bounds for \( \lambda_M \) obtained in this way will be narrow only for fairly large \( a \), the magnitude depending upon \( K(x) \). Taking the Kac case, \( K(x) = e^{-x^2} \), we obtain

\begin{equation}
2 \int_0^{a/2} e^{-x^2} dx \geq \lambda_M \geq 2 \int_0^a e^{-x^2} dx - \frac{1}{a} + \frac{e^{-a^2}}{a}
\end{equation}

which yields the results

\begin{align*}
.843 \geq \lambda_M(2)/\pi^{1/2} & \approx .713, \\
.995 \geq \lambda_M(4)/\pi^{1/2} & \approx .749, \\
.999 \approx \lambda_M(10)/\pi^{1/2} & \approx .899.
\end{align*}

Notice that even for small \( a \), (10) yields a rough idea of the true value of \( \lambda_M \).

3. **Second proof.** The method we present below yields the following useful result:

**Theorem 2.** If \( K(x) \) is non-negative, continuous, even, and monotone decreasing for \( 0 \leq x < \infty \), the characteristic function \( f_M(x) \) associated with \( \lambda_M \), which we normalize by the requirement that \( \int_0^a f_M(x) dx = 1 \), possesses the following properties:

\begin{enumerate}
\item \( f_M(x) = f_M(a - x) \),
\item \( f_M \) is monotone increasing in \( 0 \leq x \leq a/2 \).
\end{enumerate}

**Proof.** We require the following two lemmas, the first of which is a well known result in the theory of integral equations:

**Lemma 3.** Let \( K(x, y) \) be a continuous symmetric function defined over the square \( 0 \leq x, y \leq a \), and \( g(x) \) be continuous over \( 0 \leq x \leq a \). Then, if we define

\begin{equation}
Tg = \int_0^a K(x, y) g(y) dy,
\end{equation}

the limit

\begin{equation}
\lim_{n \to \infty} \frac{T^n g}{\lambda^n_M} = \phi(x)
\end{equation}
exists and is a characteristic function of \( K(x, y) \) associated with \( \lambda_M \), provided that it is not identically zero.

**Lemma 4.** If \( f(x) \) has the following properties:

1. \( f(x) = f(a - x) \),
2. \( f'(x) \geq 0 \) for \( 0 \leq x \leq a/2 \),
3. \( f(0) \geq 0 \),

then

\[
Tf = \int_0^a K(x - y)f(y)dy
\]

possesses the same properties, provided that \( K(x) \) is even, non-negative, monotone decreasing in the interval \( [0, a] \), and possesses a derivative in this interval.

**Proof of Lemma 4.** We have

\[
g(x) = Tf = 2 \int_0^{a/2} [K(x - y) + K(a - x - y)]f(y)dy
\]

whence

\[
g'(x) = 2 \int_0^{a/2} [K'(x - y) - K'(a - x - y)]f(y)dy.
\]

Integration by parts yields

\[
g'(x) = 2f(0) [K(x) - K(a - x)]
\]

\[+ 2 \int_0^{a/2} [K(x - y) - K(a - x - y)]f'(y)dy.\]

If \( 0 \leq x, y \leq a/2 \), we have

\[x \leq a - x, \quad |x - y| \leq a - x - y,
\]

and consequently

\[K(x) \geq K(a - x), \quad K(x - y) \geq K(a - x - y).
\]

Therefore \( g'(x) \geq 0 \), with equality at \( x = a/2 \).

We now combine Lemmas 3 and 4 to prove Theorem 2. Let \( f_0 = 1 \), and define

\[
f_{n+1} = \int_0^a K(x - y)f_n(y)dy.
\]
From Lemma 2 it follows that each \( f_n(x) \) possesses properties 3a, b, and c, since \( f_0 \) does trivially. Lemma 3 tells us that

\[
\phi(x) = \lim_{n \to \infty} f_n(x)/\lambda_M^n
\]

is a characteristic function of \( K(x-y) \) associated with \( \lambda_M \), provided that it is not identically zero. That it is nontrivial follows from the fact that 1 as a positive function cannot be orthogonal to \( f_M(x) \) which is also positive. It follows then that \( f_M(x) \) possesses the stated properties, since there is only one characteristic function associated with \( \lambda_M \).

The monotonicity property of \( f_M(x) \) will play an important role in our second approximation technique. We shall not obtain as close a bound as before, however. Let us normalize our solution, which we know to be positive by the requirement \( \int_0^a f(x) dx = 1 \). Integrating both sides of our integral equation between 0 and \( a \) we obtain

\[
\lambda_M = \int_0^a \left[ \int_0^a K(x-y) dx \right] f(y) dy
\]

(11)

\[
= 2 \int_0^{a/2} \left[ \int_0^y K(u) du + \int_y^{a-y} K(u) du \right] f(y) dy.
\]

From (11) we derive

\[
2c = \lambda_M = 4c \int_0^{a/2} f(x) dx
\]

\[
- 2 \int_0^{a/2} \left[ \int_0^y K(u) du + \int_y^{a-y} K(u) du \right] f(y) dy
\]

\[
= 2 \int_0^{a/2} \left[ c - \int_0^y K(u) du + c \right.
\]

\[
- \int_0^{a-y} K(u) du \right] f(y) dy
\]

(12)

\[
= 2 \int_0^{a/2} \int_0^y K(u) f(y) du dy
\]

\[
+ 2 \int_0^{a/2} \int_y^{a-y} K(u) f(y) du dy \geq 0.
\]

Thus for \( Y \) in \((0, a/2)\),
\[ |\lambda_M - 2c| = 2c - \lambda_M \leq 2 \int_0^Y \int_y^\infty K(u)f(y)\,du\,dy \\
+ 2 \int_Y^{a/2} \int_y^\infty K(u)f(y)\,du\,dy \\
+ \int_0^{a/2} \int_y^\infty K(u)f(y)\,du\,dy \\
\leq 2 \int_0^Y \int_0^\infty K(u)f(y)\,du\,dy + 2 \int_Y^{a/2} \int_y^\infty K(u)f(y)\,du\,dy \\
+ 2 \int_0^{a/2} f(y)\,dy \int_y^\infty K(u)\,du \\
\leq 2c \int_0^Y f(y)\,dy + \int_0^{a/2} \int_y^\infty K(u)f(y)\,du\,dy \\
+ \int_0^{a/2} K(u)\,du \\
= 2c \int_0^Y f(y)\,dy + \int_0^\infty K(u)\,du + \int_0^{a/2} K(u)\,du. \tag{13} \]

The original estimate of the authors involved 8c in place of 2c above. The simplification is due to the referee, whom we wish to thank for this and other helpful observations.

It remains to choose \( F \) advantageously and estimate \( \int_0^Y f(y)\,dy \). We have for \( 0 \leq y \leq a/2 \), using the monotonic character of \( f(x) \),

\[ \frac{1}{2} \int_0^{a/2} f(x)\,dx \leq \int_y^{a/2} f(x)\,dx \leq f(y) \left( \frac{a}{2} - y \right), \tag{14} \]

and hence \( f(y) \leq 1/(a - 2y) \). Therefore

\[ \int_0^Y f(y)\,dy \leq Y/(a - 2Y). \tag{15} \]

If \( Y \to \infty \) in such a way that \( Y/a \to 0 \) as \( a \to \infty \), we see that \( \lambda_M \to 2c \). Choosing \( Y \) so that \( 2cY/(a - 2Y) = \int_0^\infty K(u)\,du \), we obtain a best possible error term from this procedure. For example, if \( K(x) = e^{-x^2} \), we obtain as \( a \to \infty \)

\[ |\lambda_M - 2c| = O \left( \frac{(\log a)^{1/2}}{a} \right) \tag{16} \]

which is inferior to the result stated in Theorem 1.
4. An approximation method for small $a$. Referring to (5) of §2, we see that it is possible to improve our estimates for $\lambda_M$ by choosing, in place of $f=g=1$, functions which more nearly represent $f_M(x)$. Since we know the general form of $f_M(x)$ from Theorem 2, it would seem that two classes of functions which might yield good results are

(1) \[ f(x) = 1 + cx(a - x), \quad c \geq 0, \]

and

(2) \[
\begin{align*}
    f(x) &= 1, \quad 0 \leq x \leq b < a/2, \\
    &= c, \quad b \leq x \leq a - b, \\
    &= 1, \quad a - b \leq x \leq a, \quad c \geq 1.
\end{align*}
\]

In each of these cases the numerical work connected with approximating to the largest characteristic root of the kernel $e^{-x-y}$ will not be overly complicated since all the integrals that occur may be evaluated in terms of known functions.

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