A NOTE ON NONCOMMUTATIVE POLYNOMIALS

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1. Introduction. We shall say that an integral domain\(^1\) \(R\) satisfies condition (M) if any two nonzero elements of \(R\) have a nonzero common right multiple. In this note it is proved that if \(S\) is an extension of a ring \(R\) such that \(S\) is, roughly speaking, a noncommutative polynomial ring in one variable with \(R\) as a coefficient ring, and if \(R\) has the property (M), then \(S\) has property (M). In case \(R\) is a division ring, this result has been proved by Ore \([5]\).\(^2\) From our result it follows that the property (M) is preserved under an arbitrary number of extensions of the type described. It was first proved by Ore \([4]\) that the condition (M) is necessary and sufficient in order that an integral domain \(R\) have a uniquely determined right quotient division ring. Our method is applied to prove that the Birkhoff-Witt algebra \([1; 6]\) of a solvable Lie algebra over an arbitrary field of characteristic zero satisfies condition (M), and consequently has a uniquely determined right quotient division ring.\(^3\) It seems to be an unsolved problem to determine whether or not the Birkhoff-Witt algebra of an arbitrary Lie algebra satisfies condition (M).

2. Ring extensions. Let \(R\) and \(S\) be rings such that \(R \subseteq S\). We shall say that \(S\) is an extension of type \(O\) of \(R\) if the following conditions are satisfied:

\(\text{a})\) \(R\) and \(S\) have the same identity element.

\(\text{b})\) \(S\) contains an element \(x\) not in \(R\) such that for each \(r \in R\), \(rx = xT(r) + D(r),\) \(T(r), D(r) \in R\).

\(\text{c})\) \(R\) and \(x\) generate \(S\), that is, upon applying \(\text{b})\) every element of \(S\) can be written in the form \(\sum_{i=0}^{m} x^i r_i, r_i \in R\).

\(\text{d})\) \(\sum_{i=0}^{m} x^i r_i = 0\) implies \(r_i = 0, 0 \leq i \leq m\).

From assumptions \(\text{b})\) and \(\text{d})\) it follows that \(T(r+s) = T(r) + T(s),\) \(T(rs) = T(r)T(s);\) \(D(r+s) = D(r) + D(s)\) and \(D(rs) = D(r)T(s) + rD(s)\). Ore proved \([5, \text{p. 481}]\) that if \(R\) is an integral domain, and if we define the degree of an element \(\sum_{i=0}^{m} x^i r_i, r_m \neq 0\), to be \(m\), and deg \(0 = -\infty\), then the statement that \(rx = xT(r) + D(r)\) is a necessary consequence of the assumption that

\(^1\) By an integral domain we mean a noncommutative associative ring with an identity element, containing no zero divisors.

\(^2\) Numbers in brackets refer to the list of references at the end of the paper.

\(^3\) I am indebted to Professor N. Jacobson for suggesting this problem.
(1) \[ \text{deg } fg = \text{deg } f + \text{deg } g, \quad f, g \in S. \]

Conversely, we observe that if \( S \) is an extension of type \( O \) of an integral domain \( R \), then (1) holds for all \( f, g \in S \).

**Lemma.** Let \( R \) be an integral domain satisfying condition (M). If \( S \) is an extension of \( R \) of type \( O \), then \( S \) is an integral domain satisfying (M).

**Proof.** That \( S \) is an integral domain follows from (1). Next we prove that \( S \) has a modified division process, namely, that if \( f \) and \( g \) are nonzero elements of \( S \), then there exists a nonzero element \( a \in R \), and elements \( h \) and \( k \in S \) such that

(2) \[ fa = gh + k, \quad \text{deg } k < \text{deg } g. \]

If \( \text{deg } f < \text{deg } g \), then \( f = g0 + f \), satisfying (2). If \( \text{deg } f \geq \text{deg } g \), then by induction we may assume that (2) holds for all elements of \( S \) having lower degree than \( f \). Let \( f = \sum_{i=0}^{n} x^i a_i, \quad g = \sum_{j=0}^{m} x^j b_j \), where neither \( a_n \) nor \( b_m \) is equal to zero. Now \( gx^{n-m} = x^n T^{n-m}(b_m) + \cdots \), where the positive integral powers of \( T \) are defined inductively by the formula \( T^n(r) = T(T^{n-1}(r)), \ n = 2, 3, \cdots \). Since \( R \) satisfies (M), there exist nonzero elements \( c, d \in R \) such that \( ac = T^{n-m}(b_m)d \). If we set \( f_1 = fc - gx^{n-m}d \), then the terms of highest degree cancel, and \( \text{deg } f_1 < \text{deg } f \). Applying our induction hypothesis we have \( f_1 e = gh + k \), \( \text{deg } k < m \), and

\[ fce = g(x^{n-m}e + h) + k, \]

proving (2).

Now let \( f \) and \( g \) be nonzero elements of \( S \); we wish to prove that they have a nonzero common right multiple. We observe first that by applying (2) it is clearly sufficient to consider the case where \( \text{deg } f < \text{deg } g \). We prove by induction on the degree of \( g \) that \( g \) and any polynomial of degree less than \( \text{deg } g \) have a nonzero common right multiple. If \( \text{deg } g = 1 \) and if \( \text{deg } r = 0 \), that is, if \( r \in R \), then by applying (2) we have \( ga = rh + s \), \( \text{deg } s < 0 \), hence \( s = 0 \), proving that \( g \) and \( r \) have a common right multiple. We assume now that if \( \text{deg } f < \text{deg } g \), then \( f \) and any polynomial of lower degree have a nonzero common right multiple. Let \( \text{deg } f < \text{deg } g \); then by (2) we have \( gc = fp + q \), \( \text{deg } q < \text{deg } f \). By our induction hypothesis \( f \) and \( g \) have

\(^4\) The main idea in the proof of this lemma was suggested by Professor N. Jacobson. This argument replaces the author's much longer proof, which consisted in showing that \( S \) could be imbedded in a principal ideal domain, in which common multiples are known to exist, and then observing that as a consequence, \( S \) itself must satisfy (M).
a common right multiple, and hence \( f \) and \( g \) have a common right multiple as required.

**Theorem 1.** Let \( R \) be an integral domain satisfying the condition (M). Let \( S \) be a ring containing \( R \) which is the join of a well ordered sequence \( \{ S_\alpha \} \) of subrings, where \( \alpha \) runs through some set of ordinal numbers. If \( \alpha_0 \) is the least ordinal number in the set, let \( R = S_{\alpha_0} \), and if \( \alpha < \beta \), let \( S_\alpha \subseteq S_\beta \). Let \( S_\beta \) be an extension of type \( 0 \) of \( S_\alpha \) if \( \beta = \alpha + 1 \), and an extension of type \( 0 \) of \( \sum_{\alpha < \beta} S_\alpha \) if \( \beta \) is a limit ordinal. Then \( S \) is an integral domain satisfying condition (M).

**Proof.** We proceed by transfinite induction, assuming that \( S_\alpha \) satisfies the conclusion of the theorem for all \( \alpha < \beta \). It is sufficient to prove that \( S_\beta \) satisfies the conclusion of the theorem. If \( \beta = \alpha + 1 \), then upon applying the lemma to \( S_\alpha \), we have our result. If \( \beta \) is a limit ordinal, then we apply the lemma to \( \sum_{\alpha < \delta} S_\alpha \), first observing that \( \sum_{\alpha < \delta} S_\alpha \) satisfies the hypothesis of the lemma, since it is the join of an increasing sequence of subrings, each of which satisfies these conditions. Therefore \( S_\beta \) satisfies the conclusions of the theorem for all \( \beta \) and since \( S = \sum_{\delta} S_\delta \), \( S \) itself is an integral domain satisfying condition (M).

**Corollary.** Let \( S \) satisfy the hypotheses of the theorem. Then \( S \) has a uniquely determined right quotient division ring.

3. **An application.** We shall apply Theorem 1 to prove the following result.

**Theorem 2.** Let \( A \) be the Birkhoff-Witt algebra of a solvable Lie algebra \( L \) over an arbitrary field of characteristic zero. Then any two nonzero elements of \( A \) have a nonzero common right multiple.

**Corollary.** If \( A \) satisfies the hypothesis of Theorem 2, then \( A \) has a uniquely determined right quotient division ring.\(^6\)

**Proof of Theorem 2.** The Birkhoff-Witt algebra is defined in [1; 3], and in [6]. Harish-Chandra has proved [3, Corollary 1.2] that \( A \) is an integral domain. We shall prove that \( A \) satisfies the condition (M). First assume that the theorem has been proved when the base field is algebraically closed and of characteristic zero. Consider a solvable Lie algebra \( L \) over an arbitrary field \( \Phi \) of characteristic zero, and let \( A \) be its Birkhoff-Witt algebra. Let \( \Gamma \) be the algebraic closure

\(^6\) It is not difficult to prove, using the methods of [2, p. 150], that \( A \) can also be imbedded in a left quotient ring, and that the left and right quotient division rings are isomorphic.
of \( \Phi \), and let \( \{ \gamma_i \} \) be a basis for \( \Gamma \) over \( \Phi \). If we form the Kronecker product algebra \( A \otimes \Gamma \) of \( A \) and \( \Gamma \) with respect to \( \Phi \), then \( A \otimes \Gamma \) is the Birkhoff-Witt algebra of the solvable Kronecker product Lie algebra \( L \otimes \Gamma \) over the algebraically closed field \( \Gamma \), and we may assume that any two nonzero elements of \( A \otimes \Gamma \) have a common right multiple. Observe that \( A \subseteq A \otimes \Gamma \). If \( a \) and \( b \) are nonzero elements of \( A \), then there exist nonzero elements \( a_1, b_1 \in A \otimes \Gamma \) such that \( ab_1 = b_1a \).

We can write \( a_1 \) and \( b_1 \) uniquely in the form \( a_1 = \sum u_i \otimes \gamma_i, b_1 = \sum w_j \otimes \gamma_j, u_i \) and \( w_j \in A \), and we have

\[
\sum a u_i \otimes \gamma_i = \sum b w_j \otimes \gamma_j.
\]

Since these expressions are unique, for at least one index \( i \), \( a u_i = b w_i \neq 0 \). This proves that \( A \) satisfies the condition (M).

We may assume, therefore, that \( \Phi \) is algebraically closed, and of characteristic zero. It follows from one of Lie's theorems that \( L \) has a basis \( X_1, \ldots, X_n \) having the commutation rules

\[
[X_i, X_j] = \sum_k \alpha_{ijk} X_k, \quad \alpha_{ijk} \in \Phi,
\]

where \( \alpha_{ijk} = 0 \) if \( k < \min (i, j) \). With respect to this basis we construct the Birkhoff-Witt algebra \( A \) of \( L \), consisting of noncommutative polynomials in \( n \) letters \( x_1, \ldots, x_n \) subject to the commutation rules

\[
[x_i, x_j] = x_i x_j - x_j x_i = \sum_k \alpha_{ijk} x_k.
\]

It is known [6] that the standard monomials \( x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}, e_j \geq 0 \), form a basis for \( A \) over \( \Phi \). From (4) it follows that for each \( k, 1 \leq k < n \), the \( \Phi \)-subspace \( A_k \) consisting of polynomials in \( x_{n-k+1}, \ldots, x_n \) alone is closed under multiplication and hence is a subring, and that for each \( k \), the subring is sent into itself by the derivation \( a \rightarrow ax_{n-k} - x_{n-k}a = [a, x_{n-k}] \). Furthermore \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n = A \). We prove next that for each \( k < n \), \( A_{k+1} \) is an extension of type O of \( A_k \). From what has already been said, it is sufficient to verify condition (d). Let

\[
\sum_{i=0}^{r} x_{n-k}^i b_i = 0, \quad b_i \in A_k, 0 \leq i \leq r.
\]

We can write each \( b_i \) as a linear combination of the standard monomials in \( x_{n-k+1}, \ldots, x_n \) and the left side of (5) becomes a linear combination of standard monomials in \( x_{n-k}, \ldots, x_n \). Since the standard monomials are linearly independent, the coefficients of all the \( b_i \) must be zero, proving (d). Finally \( A_1 \), which is the polynomial algebra in \( x_n \) with coefficients in \( \Phi \), is a commutative integral domain,
and consequently satisfies condition (M). Applying Theorem 1 to $A$, we conclude that $A$ is an integral domain satisfying condition (M), and the proof of Theorem 2 is complete.

*Added in proof, December 12, 1952.* A proof of the fact that the Birkhoff-Witt algebra of an arbitrary Lie algebra satisfies condition (M) has been announced by D. Tamari (Bull. Amer. Math. Soc. Abstract 58-5-527.)

**References**


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