1. Introduction. Consider the continued fraction

\[ f_1 + \frac{a_1}{b_1 - a_2} \frac{1}{b_2 - a_3} \cdots \]

where \( f_1 \) is a number, \( \{a_1, a_2, a_3, \ldots\} \) is a sequence of nonzero numbers, and \( \{b_1, b_2, b_3, \ldots\} \) is a sequence of numbers. We obtain conditions necessary and sufficient for (1.1) to converge absolutely, and we indicate their relationship to older sufficient conditions. We find a new characterization of positive definite continued fractions, whose importance is emphasized by the fact (Theorem 4.2) that if (1.1) converges, then there is a positive definite continued fraction which is a contraction of (1.1). We also obtain new sufficient conditions for absolute convergence of positive definite continued fractions.

2. Continued fractions and sequences of linear fractional transformations. In this paper, a subscript \( p \) denotes a positive integer. By the generator of (1.1) we mean the sequence \( \{t_1(u), t_2(u), t_3(u), \ldots\} \) of linear fractional transformations such that \( t_1(u) = f_1 + a_1/(b_1 - u) \) and \( t_{p+1}(u) = t_p[a_{p+1}/(b_{p+1} - u)] \) for \( p \geq 1 \). We denote this sequence by \( t(u) \).

Remark 2.1. For a sequence \( s(u) \) of linear fractional transformations to be the generator of a continued fraction, it is necessary and sufficient that \( s_1(\infty) \neq \infty \) and \( s_p(0) = s_{p+1}(\infty) \) for \( p \geq 1 \).

By the sequence of approximants of (1.1) we mean the sequence \( \{f_1, f_2, f_3, \ldots\} \) such that \( f_p = t_p(\infty) \) for \( p \geq 1 \). We denote this sequence by \( f \).

Remark 2.2. For a sequence \( x \) of points in the complex plane to be the sequence of approximants of a continued fraction with nonzero partial numerators, it is necessary and sufficient that \( x_1 \neq \infty \) and \( x_p \neq x_{p+1} \) for \( p \geq 1 \).
If \( f \) has the property that there exists a positive integer \( n \) such that (1) the sequence \( \{f_n, f_{n+1}, f_{n+2}, \ldots \} \) is bounded and (2) either \( n = 1 \) or \( f_{n-1} = \infty \), then by \( B_f \) we mean the set of all sequences \( R \) such that for \( p \geq 1 \)

\[
\begin{align*}
1) & \quad R_p \text{ is a circle plus its interior,} \\
2.1) & \quad R_p \supseteq R_{p+1}, \text{ and} \\
3) & \quad f_p \text{ is in } R_p \text{ if } p \geq n.
\end{align*}
\]

**Theorem 2.1.** If \( f \) is bounded, then for \( R \) to be a member of \( B_f \) it is necessary and sufficient that

\[
\begin{align*}
1) & \quad R_1 \text{ is a circle plus its interior,} \\
2) & \quad \text{if } p \geq 1, \text{ then } t_p^{-1}(R_p) \text{ is a closed half-plane or a circle plus its exterior, and} \\
3) & \quad \text{if } p \geq 1, \text{ then } t_p^{-1}(R_p) \supseteq t_{p+1}^{-1}(R_{p+1}).
\end{align*}
\]

Moreover, if \( R \) is a sequence in \( B_f \), and if \( p \geq 1 \), then \( t_p^{-1}(R_p) \) is a closed half-plane if \( f_p \) is a boundary point of \( R_p \), or is a circle plus its exterior if \( f_p \) is an interior point of \( R_p \).

**Proof.** The theorem is a direct consequence of the definitions of \( f \), \( t(u) \), and \( B_f \).

We denote by \( h \) the sequence \( \{h_1, h_2, h_3, \ldots \} \) of points in the complex plane such that if \( p \geq 1 \) then

\[
h_p = t_{p+1}^{-1}(\infty).
\]

From the relations \( t_1(b_1) = \infty \) and \( t_{p+1}(u) = t_p[a_{p+1}/(b_{p+1} - u)] \), it follows that

\[
h_1 = b_1 \quad \text{and} \quad h_{p+1} = b_{p+1} - a_{p+1}/h_p \quad \text{for } p \geq 1.
\]

If \( p \geq 1 \), then

\[
t_p(\infty) = f_p, \quad t_p(0) = f_{p+1}, \quad \text{and} \quad t_{p+1}(b_{p+1}) = f_p;
\]

so that

\[
f_p = \infty \text{ if and only if } h_p = \infty,
\]

\[
f_{p+1} = \infty \text{ if and only if } h_p = 0, \text{ and}
\]

\[
f_p = \infty \text{ if and only if } h_{p+1} = b_{p+1}.
\]

If \( p \geq 1 \), and if \( f_p \neq \infty \) and \( f_{p+1} \neq \infty \), then

\[
t_p(u) = f_p + \frac{h_p(f_{p+1} - f_p)}{h_p - u}.
\]

If \( p \geq 1 \), and if \( f_p \neq \infty \), \( f_{p+1} \neq \infty \), and \( f_{p+2} \neq \infty \), then

\[
\frac{f_{p+1} - f_{p+2}}{f_p - f_{p+1}} = \frac{a_{p+1}}{h_{p+1}} = \frac{a_{p+1}}{h_p h_{p+1}}.
\]

3. **Conditions necessary and sufficient for absolute convergence.**

If \( x \) is a sequence of points in the complex plane, the statement that \( x \) converges absolutely means that there exists a positive integer \( n \)
such that (1) if \( p \geq n \), then \( x_p \neq \infty \) and (2) \( \sum_{p=n}^{\infty} |x_p - x_{p+1}| \) converges. The statement that a continued fraction converges absolutely means that its sequence of approximants converges absolutely.

**Theorem 3.1.** For (1.1) to converge absolutely, it is necessary and sufficient that there exist a positive integer \( n \), a sequence \( s \) of numbers, and a sequence \( q \) of numbers such that

\[
\begin{align*}
(1) & \quad s_p > 0 \text{ and } q_p \neq 0 \text{ for } p \geq n, \text{ and } \sum_{p=n}^{\infty} s_p \text{ converges,} \\
(2) & \quad \text{there is a sequence } R \text{ in } B_f \text{ such that if } p \geq n, \text{ then } t_p^{-1}(R_p) \text{ is the region defined by the inequality } s_p \cdot |u| \leq |u - q_p|, \text{ and} \\
(3) & \quad \text{there is a sequence } R' \text{ in } B_f \text{ such that if } p \geq n, \text{ then } q_p \text{ is in } t_p^{-1}(R'_p). 
\end{align*}
\]

**Proof.** A. Suppose that there exist such an integer \( n \) and such sequences \( s \) and \( q \). Let \( m \) denote an integer such that if \( p = m \), then \( p \geq n \) and \( f_p \) is in \( R_p \). Now \( R_p \) is a circle plus its interior, \( \infty \) is not in \( R_p \), and \( h_p = t_p^{-1}(\infty) \) is not in \( t_p^{-1}(R_p) \); hence if \( p \geq m \), then \( s_p \cdot |h_p| > |h_p - q_p| \), or \( s_p > |(h_p - q_p)/h_p| \). Moreover, if \( p \geq m \), then by (2.4) and (2.5),

\[
|f_{p+1} - f_p|/(t_p(q_p) - f_p) = |(h_p - q_p)/h_p| < s_p.
\]

By hypothesis, \( f_p \) is in \( R_m \) and \( t_p(q_p) \) is in \( R'_m \), and consequently there exists a number \( M \) such that if \( p \geq m \), then \( t_p(q_p) - f_p < M \), so that \( |f_{p+1} - f_p| < Ms_p \). Since \( \sum_{p=n}^{\infty} s_p \) converges, (1.1) converges absolutely.

B. Suppose that (1.1) converges absolutely. Let \( n \) denote the positive integer such that if \( p \geq n \), then \( f_p \neq \infty \) and such that either \( n = 1 \) or \( f_{n-1} = \infty \). Let \( R_n \) be a circle plus its interior, with radius \( r \) and center \( c \) such that if \( p \geq n \), then \( 3r > 4|f_p - c| > 2r \). Let \( R'_n \) be a circle plus its interior with radius \( r' \) and center \( c \), such that \( R'_n \supset R_n \) and such that if \( p \geq n \), then the inversion of \( f_p \) in the boundary of \( R_n \) is in \( R'_n \). For \( p \geq 1 \), let \( R_p = R_n \) and \( R'_p = R'_n \). Then \( R \) is in \( B_f \) and \( R' \) is in \( B_f \).

For \( p \geq n \), let \( t_p(q_p) \) be the inversion of \( f_{p+1} \) in the boundary of \( R_p \). By construction, \( t_p(q_p) \) is in \( R'_p \), so that \( q_p \) is in \( t_p^{-1}(R'_p) \). Moreover, if \( p \geq n \), then there exists a positive number \( s'_p \) such that \( R_p \) is the region defined by \( s'_p |u - f_{p+1}| \leq |u - t_p(q_p)| \); and since \( 3r > 4|f_{p+1} - c| > 2r \), there exist positive numbers \( D \) and \( s' \) such that \( |t_p(q_p) - f_p| \leq D \) and \( s'_p \leq s' \) for \( p \geq n \). By (2.5), \( t_p^{-1}(R_p) \) is the region defined by \( s_p |u| \leq |u - q_p| \), where \( s_p = s'_p |(f_{p+1} - f_p)/(t_p(q_p) - f_p)| < |f_{p+1} - f_p|s'/D \). Hence \( \sum_{p=n}^{\infty} s_p \) converges. This completes the proof.

**Lemma 3.2a.** If \( s \) is a sequence of positive numbers, then for \( \sum_{p=1}^{\infty} s_p \) to converge it is necessary and sufficient that there exist a sequence \( d \) of
positive numbers such that for $p \geq 1$

$$\frac{s_{p+1}}{s_p} \leq \frac{d_p}{1 + d_{p+1}}.$$

**Proof.** Suppose that $d$ is such a sequence. If $p \geq 1$, then $s_{p+1} \leq s_p$, and by induction, if $n$ is an integer greater than $p$, then $\sum_{k=1}^n s_k \leq s_p d_p$, so that $\sum_{k=1}^n s_k < \sum_{k=1}^p s_p + s_p d_p$. Hence $\sum_{k=1}^n s_k$ converges.

Suppose that $\sum_{k=1}^n s_k$ converges. Let $r$ be a sequence of non-negative real numbers such that $\sum_{k=1}^n r_k$ converges, and for $p \geq 1$, let $d_p$ be the positive number such that $s_p d_p = \sum_{k=1}^n (r_k + s_k)$. Then $s_p d_p = r_{p+1} + s_{p+1} d_{p+1} \leq s_{p+1} + s_{p+1} d_{p+1}$, so that $s_{p+1}/s_{p} \leq d_p/(1 + d_{p+1})$. This completes the proof.

**Remark 3.1.** From the above proof it follows that if in Lemma 3.2a the statement $s_{p+1}/s_p \leq d_p/(1 + d_{p+1})$ is replaced by either of the statements

$$\frac{s_{p+1}}{s_p} < \frac{d_p}{1 + d_{p+1}}, \quad \frac{s_{p+1}}{s_p} = \frac{d_p}{1 + d_{p+1}},$$

then the resulting lemma is true.

**Example 3.1.** Let $a > -1$, $b > a+1$, and $d_p = (a+p)/(b-a-1)$ for $p \geq 1$. By Lemma 3.2a, the series

$$1 + \frac{a + 1}{b + 1} + \frac{(a + 1)(a + 2)}{(b + 1)(b + 2)} + \cdots$$

converges.

**Lemma 3.2b.** For $f$ to converge absolutely, it is necessary and sufficient that there exist a positive integer $n$ and a sequence $d$ of positive numbers such that, for $p \geq n$,

(i) $d_p > 1 + d_{p+1}$ if $f_{p+1} = \infty$ or if $f_p = f_{p+2} = \infty$ and

(ii) $d_p |f_p - f_{p+1}| > (1 + d_{p+1}) |f_{p+1} - f_{p+2}|$ if

(3.2)

(a) $f_{p+1} \neq \infty$ and

(b) $f_p \neq \infty$ or $f_{p+2} \neq \infty$.

**Proof.** If $f$ converges absolutely, then there exists a positive integer $n$ such that $f_p \neq \infty$ if $p \geq n$; and by Remark 3.1 there exists a sequence $d$ of positive numbers such that (ii) holds for $p \geq n$.

Suppose that there exist a positive integer $n$ and a sequence $d$ of positive numbers such that (3.2) holds for $p \geq n$. We first show that if $p \geq n + d_n$, then $f_p \neq \infty$. Suppose that $m$ is an integer, that $m \geq n + d_n$, and that $f_m = \infty$. Then for $p = m - 1$, the relation (i) holds by hypothesis, and $d_{m-1} > 1 + d_m > 1$. Since $f_m = \infty$, it follows (Remark
(2.2) that \( f_{m-1} \neq \infty \). If \( f_{m-2} \neq \infty \), therefore, (ii) must hold for \( p = m-2 \); but this is impossible, since \( f_m = \infty \). Hence \( f_{m-2} = \infty \), and (i) holds for \( p = m-2 \), so that \( d_{m-2} > 1 + d_{m-1} > 2 \). If \( m > n+2 \), then (i) must hold for \( p = m-3 \), and \( d_{m-3} > 3 \). If \( m > n+3 \), then \( f_{m-4} = \infty \) and \( d_{m-4} > 4 \). By induction, \( d_n > m-n \), so that \( m < n + d_n \). Hence the assumption that \( f_m = \infty \) is false; and if \( p \geq n + d_n \), then \( f_p \neq \infty \). By Lemma 3.2a, \( f \) converges absolutely. This completes the proof.

**Theorem 3.2.** For (1.1) to converge absolutely, it is necessary and sufficient that there exist a positive integer \( n \) and a sequence \( d \) of positive numbers such that, for \( p \geq n \),

\[
\begin{align*}
(i) & \quad \text{if } b_{p+1} = 0, \text{ then } d_p > 1 + d_{p+1}, \\
(ii) & \quad \text{if } b_{p+1} \neq 0 \text{ and if } t_{p+1}^{-1}(K_{p+1}) \text{ is the region defined by } d_p |u| \leq (1 + d_{p+1}) |u - b_{p+1}|, \text{ then } K_{p+1} \text{ is a circle plus its interior.}
\end{align*}
\]

**Proof.** The conditions (3.2) of Lemma 3.2b can be written

\[
\begin{align*}
(a) & \quad d_p > 1 + d_{p+1} \text{ if } f_p = f_{p+2}, \\
(b) & \quad d_p > 1 + d_{p+1} \text{ if } f_p \neq f_{p+2} \text{ and } f_{p+1} = \infty, \\
(c) & \quad d_p |f_p - f_{p+1}| > (1 + d_{p+1}) |f_{p+1} - f_{p+2}| \text{ if } f_p \neq f_{p+2} \text{ and } f_{p+1} \neq \infty.
\end{align*}
\]

Since \( f_p = t_{p+1}(b_{p+1}), f_{p+2} = t_{p+1}(0), \) and \( \infty = t_{p+1}(h_{p+1}) \), the first two of these conditions can be written

\[
\begin{align*}
(a') & \quad d_p > 1 + d_{p+1} \text{ if } b_{p+1} = 0, \\
(b') & \quad d_p > 1 + d_{p+1} \text{ if } b_{p+1} \neq 0 \text{ and } h_{p+1} = \infty;
\end{align*}
\]

as for the third, where \( b_{p+1} \neq 0 \) and \( h_{p+1} \neq \infty \), similar consideration of the two cases (1) \( h_{p+1} = 0 \) and (2) \( h_{p+1} = b_{p+1} \), and use of (2.6) for the case (3) \( h_{p+1} \neq 0, h_{p+1} \neq b_{p+1} \), shows that (c) may be written

\[
(c') d_p |h_{p+1} - b_{p+1}| > (1 + d_{p+1}) |h_{p+1} - b_{p+1}| \text{ if } b_{p+1} \neq 0 \text{ and } h_{p+1} \neq \infty.
\]

Now if \( K_{p+1} \) is defined by \( d_p |u| \leq (1 + d_{p+1}) |u - b_{p+1}| \), where \( d_p > 0, d_{p+1} > 0, \) and \( b_{p+1} \neq 0 \), then for \( K_{p+1} \) to be a circle plus its interior, it is necessary and sufficient that the point \( h_{p+1} = t_{p+1}^{-1}(\infty) \) be exterior to \( t_{p+1}^{-1}(K_{p+1}) \). Hence the conditions \( (a'), (b'), \) and \( (c') \) are equivalent to (3.3), and the theorem now follows from Lemma 3.2b. This completes the proof.

**Remark 3.2.** If \( a_1 = 1 \) and \( b_1 = 1 \) and \( a_{p+1} = -c_p \) for \( p \geq 1 \), then (1.1) is the continued fraction

\[
\frac{1}{1 + c_1} \frac{1}{1 + c_2} \frac{1}{1 + \cdots}.
\]
where $c$ is a sequence of nonzero numbers. If, in the notation of Theorem 3.2, $r_p = d_p/(1 + d_{p+1})$, then $t^{-1}_{p+1}(K_{p+1})$ is defined by the inequality $r_p |u| \leq |u - 1|$. The condition $t^{-1}_p(K_p) \supseteq t^{-1}_p(K_{p+1})$ gives the inequalities (5.5), p. 376, of Lane and Wall [1] for $p \geq 1$. The condition $t^{-1}_p(K_{p-1}) \subseteq t^{-1}_p(K_{p-1}p+1)$ gives, for $p \geq 2$, the inequalities (13) of Scott and Wall [2].

4. A characterization of positive definite continued fractions. The continued fraction (1.1) is said to be positive definite if

(i) $I(b_1) > 0$ and $I(b_p) \geq 0$ for $p > 1$, and

(ii) there exists a sequence $g$ of numbers such that $0 < g_1 \leq 1$

and, for $p \geq 1$, $0 \leq g_{p+1} \leq 1$ and

$$|a_{p+1} - R(a_{p+1})| \leq 2I(b_p)I(b_{p+1})(1 - g_p)g_{p+1}.$$ 

If $F$ is a continued fraction, the statement that $F$ is equivalent to (1.1) means that the sequence of approximants of $F$ is the sequence of approximants of (1.1).

Remark 4.1. If $F$ is a continued fraction, and if $t'(u)$ is the generator of $F$, then for $F$ to be equivalent to (1.1) it is necessary and sufficient that there exist a sequence $\sigma$ of nonzero numbers such that $t'_p(u) = t_p(u/\sigma_p)$ for $p \geq 1$. If $\sigma$ is such a sequence, then $F$ is the continued fraction

$$f_1 + \frac{\sigma_1 a_1}{\sigma_1 b_1 - \sigma_1 \sigma_2 a_2} + \frac{\sigma_2 b_2 - \sigma_2 \sigma_3 a_3}{\sigma_3 b_3 - \cdots}.$$ 

Theorem 4.1. For (1.1) to be equivalent to a positive definite continued fraction, it is necessary and sufficient that there exist a sequence $R$ in $B_f$ such that if $p \geq 1$, then $t^{-1}_p(R_p)$ is a closed half-plane; i.e., if $p \geq 1$, then $f_p$ is a boundary point of $R_p$.

Proof. Let $t^{-1}(R)$ be a sequence of closed half-planes. Then there exist a sequence $\sigma$ of nonzero numbers and a sequence $k$ of real numbers such that if $p \geq 1$, then $t^{-1}_p(R_p)$ is defined by $R(\sigma_p u) \leq k_p$. We show first that for $R$ to be in $B_f$ it is necessary and sufficient that

(i) $0 \leq k_1 < R(\sigma_1 b_1)$ and $0 \leq k_p \leq R(\sigma_p b_p)$ for $p > 1$, and

(ii) $R(\sigma_p a_{p+1}a_{p+1}) + |\sigma_p a_{p+1}a_{p+1}| \leq 2k_p R(\sigma_p b_{p+1}b_{p+1} - k_{p+1})$ for $p \geq 1$.

Numbers in brackets refer to the bibliography at the end of the paper.

This is an adaptation to (1.1) of the definition on pp. 67–71 of [3], where it is assumed that $g_i I(b_i) > 0$; e.g., in formula (17.3) of [3].
For $R_1$ to be a circle plus its interior, it is necessary and sufficient that the point $t^{-1}_1(\infty) = b_1$ be exterior to $t^{-1}_1(R_1)$; i.e., that $R(\sigma_1 b_1) > k_1$.

If $p \geq 1$, then $\infty$ is a boundary point of $t^{-1}_p(R_p)$, and $f_p = t_p(\infty)$ is a boundary point of $R_p$; similarly, $f_{p+1}$ is a boundary point of $R_{p+1}$. If $R_p \supseteq R_{p+1}$, then the point $t^{-1}_p(f_{p+1}) = 0$ is in $t^{-1}_p(R_p)$, or $0 \leq k_p$; moreover, the point $t^{-1}_{p+1}(f_p) = b_{p+1}$ is not an interior point of $t^{-1}_{p+1}(R_{p+1})$, or $R(\sigma_{p+1} b_{p+1}) \leq k_{p+1}$. Hence for $t^{-1}(R)$ to be in $B_f$, the conditions (i) of (4.2) are necessary.

Suppose that (i) of (4.2) holds. Then for $p \geq 1$, $t^{-1}_p(R_{p+1})$ is defined by

$$R(\sigma_{p+1} a_{p+1} u) \geq 0 \quad \text{if} \quad R(\sigma_{p+1} b_{p+1}) = k_{p+1},$$

and

$$u - \frac{\sigma_{p+1} a_{p+1}}{2R(\sigma_{p+1} b_{p+1} - k_{p+1})} \leq \frac{\sigma_{p+1} a_{p+1}}{2R(\sigma_{p+1} b_{p+1} - k_{p+1})}$$

if $R(\sigma_{p+1} b_{p+1}) > k_{p+1}$.

Hence if (i) of (4.2) holds, then (ii) is a condition necessary and sufficient for the relations $R_p \supseteq R_{p+1}$ to hold for $p \geq 1$. We conclude that $t^{-1}(R)$ is in $B_f$ if and only if (4.2) holds.

If for $p \geq 1$ we take $\sigma_p = -i$ and $k_p = (1-g_p)R(-ib_p)$, where $g_p = 1$ if $k_p = 0$, the theorem now follows from (4.1) and Remark 4.1. This completes the proof.

**Remark 4.2.** By Theorem 4.1, a bounded increasing infinite sequence of real numbers is the sequence of approximants of a positive definite continued fraction. More generally, if $x$ is a sequence of numbers, if $x_p \neq x_{p+1}$ for $p \geq 1$, and if there exists a number $c$ such that $|x_p - c| \geq |x_{p+1} - c|$ for $p \geq 1$, then $x$ is the sequence of approximants of a positive definite continued fraction.

**Theorem 4.2.** If (1.1) converges, then there exists a positive definite continued fraction whose sequence of approximants is a subsequence of $f$.

**Proof.** Let $c$ be the number such that $f_p \to c$ as $p \to \infty$. Then there exists an infinite subsequence, $x$, of $f$ such that if $p \geq 1$, then $x_p \neq \infty$ and $|x_p - c| > |x_{p+1} - c|$. By Remark 4.2, $x$ is the sequence of approximants of a positive definite continued fraction. This completes the proof.

5. Absolute convergence of positive definite continued fractions.

Throughout this section we suppose that (1.1) is equivalent to a positive definite continued fraction, and that $k$ is a sequence of real numbers such that $R$ is in $B_f$, where, for $p \geq 1$, $t^{-1}_p(R_p)$ is the closed half-plane $R(u) \leq k_p$. The conditions (4.2) hold, therefore, with $\sigma_p = 1$, and $t^{-1}_p(R_{p+1})$ is the region defined by (4.3), for $p \geq 1$. 

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Theorem 5.1. If there exist a positive integer \( n \) and a positive number \( M \) such that \(|a_{p+1}| \leq Mk_pR(b_{p+1} - k_{p+1})\) for \( p \geq n \), then (1.1) converges absolutely.

Proof. Since, by hypothesis, (1.1) is equivalent to a positive definite continued fraction, its sequence of approximants is bounded; and by (2.4), \( h_p \neq \infty \) and \( h_p \neq 0 \) for \( p \geq 1 \). Moreover, if \( p \geq n \), then \( t_p^{-1}(R_{p+1}) \) is a circle plus its interior; let \( v_p \) be the point of \( t_p^{-1}(R_{p+1}) \) farthest from \( h_p \). By (2.5),

\[
\frac{|f_{p+1} - f_p|}{t_p(v_p) - f_p} = \frac{|h_p - v_p|}{h_p} \leq 1 + \frac{|v_p|}{h_p}.
\]

Since the origin is a boundary point of \( t_p^{-1}(R_{p+1}) \), \( |v_p| \) is less than or equal to the diameter of \( t_p^{-1}(R_{p+1}) \), or \(|v_p| \leq |a_{p+1}|/R(b_{p+1} - k_{p+1})\); hence \( |v_p| < Mk_p \). Since \( R_p \) is a circle plus its interior, \( h_p \) is not in the closed half-plane \( R(u) \leq k_p \); so \( |h_p| \geq R(h_p) > k_p \). Finally, by (2.5), \( t_p(v_p) \) is the point of \( R_{p+1} \) nearest \( f_p \); so \(|t_p(v_p) - f_p| \leq 2(r_p - r_{p+1})\), where for \( p \geq 1 \), \( r_p \) is the radius of \( R_p \). We now conclude that if \( p \geq n \), then \(|f_{p+1} - f_p| < 2(1 + M)(r_p - r_{p+1})\). Since \( \sum_{p=n}^{\infty} (r_p - r_{p+1}) \) is a convergent positive-term series, (1.1) converges absolutely. This completes the proof.

Corollary 5.1a. If there exist a sequence \( g \) and a positive number \( M \) such that, for \( p \geq 1 \),

(i) \( 0 < g_p < 1 \),
(ii) \( |c_p| - R(c_p) \leq 2(1 - g_p)g_{p+1} \), and
(iii) \( |c_p| < M(1 - g_p)g_{p+1} \),

then the continued fraction (3.4) converges absolutely.

Remark 5.1. The above corollary is a true generalization of the convergence condition \(|c_p| \leq (1 - g_p)g_{p+1} \), of Pringsheim [4]; compare it with the condition \(|c_p| - R(c_p) \leq 2r(1 - g_p)g_{p+1} \), where \( 0 < r < 1 \), \( p \geq 1 \), on pp. 142–143 of [3].

Remark 5.2. It should be noted that in Theorem 5.1 and its corollary we do not conclude that the common part of \( R_1, R_2, R_3, \ldots \) is a point. Actually there exists an absolutely convergent positive definite continued fraction which has the property that if \( t^{-1}(R) \) is a sequence of closed half-planes such that \( R \) is in \( B_f \), then the common part of \( R_1, R_2, R_3, \ldots \) is a circle plus its interior. We give the following example. Let \( s \) be a decreasing sequence of positive numbers such that \( \sum_{p=1}^{\infty} s_p \) converges. For \( p \geq 1 \), let each of \( R_{3p-2}, R_{3p-1}, \) and \( R_p \) be the region defined by \(|u - (s_p - 1)| \leq 1 + s_p\), and let \( f_{3p-2}, f_{3p-1}, \)
and \( f_{3p} \) be boundary points of \( R_{3p} \) such that \( \arg f_{3p-2} = 0 \), \( \arg f_{3p-1} = 1 \), and \( \arg f_{3p} = -1 \). Then \( f \) is the sequence of approximants of an absolutely convergent positive definite continued fraction. If \( R' \) is a sequence in \( B_j \) such that \( f_p \) is a boundary point of \( R'_p \) for \( p \geq 1 \), then \( R'_{3p-2} \supset R_{3p-2} \) for \( p \geq 1 \), and hence the common part of \( R'_1, R'_2, R'_3, \ldots \) is a circle plus its interior.

**Theorem 5.2.** Let \( e_p = \left| a_{p+1} \right| \left[ 2k_p R(b_{p+1} - k_{p+1}) - R(a_{p+1}) \right] \) for \( p \geq 1 \). If \( \sum_{p=1}^\infty (1 - e_p) \) diverges, then (1.1) converges. If there exists a sequence \( d \) of positive numbers such that \( e_p(2 + 2d_{p+1} - d_p) \leq d_p \) for \( p \geq 1 \), then (1.1) converges absolutely.

**Proof.** A. We show first that \( r_{p+1}/r_p \leq 2e_p/(1 + e_p) \) for \( p \geq 1 \), where \( r_p \) is the radius of \( R_p \). If \( R(b_{p+1}) = k_{p+1} \), then by (4.2) \( a_{p+1} < 0 \) and hence \( e_p = 1 \), so that the relation \( r_{p+1}/r_p \leq 2e_p/(1 + e_p) \) holds. If \( R(b_{p+1}) > k_{p+1} \), then \( t_p^{-1}(R_{p+1}) \) is a circle plus its interior; let \( v_p \) be the point of \( t_p^{-1}(R_{p+1}) \) farthest from \( h_p \), and let \( w_p \) be the point of \( t_p^{-1}(R_{p+1}) \) nearest \( h_p \). By (2.5), \( t_p(v_p) \) is the point of \( R_{p+1} \) nearest \( f_p \), and \( t_p(w_p) \) is the point of \( R_{p+1} \) farthest from \( f_p \); so \( 2r_{p+1} = |t_p(v_p) - t_p(w_p)| \) and \( 2r_p \geq |t_p(w_p) - f_p| \). By (2.5), \( |t_p(v_p) - t_p(w_p)|/|t_p(w_p) - f_p| = |(v_p - w_p)/(h_p - v_p)| \); hence \( r_{p+1}/r_p \leq |(v_p - w_p)/(h_p - v_p)| \). Since the diameter of \( t_p^{-1}(R_{p+1}) \) is \( |v_p - w_p| = \left| a_{p+1} \right|/R(b_{p+1} - k_{p+1}) \), and since the distance from \( h_p \) to \( v_p \) is \( |h_p - v_p| > k_p + \left| a_{p+1} \right| - R(a_{p+1}) / 2R(b_{p+1} - k_{p+1}) \), it follows that

\[
\frac{r_{p+1}}{r_p} \leq \frac{2}{2k_p R(b_{p+1} - k_{p+1}) - R(a_{p+1}) + \left| a_{p+1} \right|} = \frac{2e_p}{1 + e_p}.
\]

B. Suppose that \( \sum_{p=1}^\infty (1 - e_p) \) diverges. Now by definition and by (4.2), \( 0 < e_p \leq 1 \); so \( \sum_{p=1}^\infty (1 - e_p)/(1 + e_p) \) diverges. But \( 1 - r_{p+1}/r_p \geq 1 - 2e_p/(1 + e_p) = (1 - e_p)/(1 + e_p) \). Hence if for \( p \geq 1 \), \( s_p = 1 - r_{p+1}/r_p \), then \( \sum_{p=1}^\infty s_p \) is a divergent series whose terms are non-negative real numbers. Since \( r_{p+1} = r_1(1 - s_1)(1 - s_2) \cdots (1 - s_p) \), it follows that \( r_p \to 0 \) as \( p \to \infty \), and consequently (1.1) converges.

C. Suppose that there exists a sequence \( d \) of positive numbers such that \( e_p(2 + 2d_{p+1} - d_p) \leq d_p \) for \( p \geq 1 \). Then for \( p \geq 1 \), \( r_{p+1}/r_p \leq 2e_p/(1 + e_p) \leq d_p/(1 + d_{p+1}) \); and by Lemma 3.2a, \( \sum_{p=1}^\infty r_p \) converges. Since \( |f_{p+1} - f_p| \leq 2r_p \), (1.1) converges absolutely. This completes the proof of the theorem.

**Example 5.1.** Let \( s \) be a positive number greater than 4. If \( 0 < e_p \leq p/s \) for \( p \geq 1 \), then the continued fraction (3.4) converges absolutely. This can be seen by taking \( k_p = 1/2 \) and \( d_p = 4p/(s - 4) \) in Theorem 5.2. In Corollary 6.1a, p. 380, of [1], it was required that an infinite subsequence of \( c \) be bounded.
Remark 5.3. If $c_p = p(p+x)/(1+x)^2$ for $p \geq 1$, it can be shown that (3.4) converges absolutely for $x > 0$; it was shown on p. 379 of [1] that if $x = 0$, then (3.4) converges but does not converge absolutely.

Bibliography


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