KERNEL FUNCTIONS AND CLASS $L^2$

I. EDWARD BLOCK

1. Introduction. Let $f(z)$ be a measurable function of the real variables $x$ and $y$ in the plane of the complex $z(=x+iy)$ variable. Denote by $B$ a finite domain bounded by a finite number of analytic Jordan curves, and by $L^2(B)$ the Hilbert space of functions $f(z)$ of summable square on $B$, i.e.,

$$
\|f\|^2_B = \int_B \int_B dxdy |f(z)|^2.
$$

The subspace of functions of $L^2(B)$ which are analytic on $B$ is designated by $L^2(B)$. Associated with $B$ is the classical reproducing kernel $K(z, \xi^*)$ [1], and it is well known that if $f_1(z)$ belongs to $L^2(B)$, then

$$
(1.1) \quad f_1(z) = \int_B \int_B d\xi d\eta K(z, \xi^*) f_1(\xi) \quad (\xi = \xi + i\eta).
$$

(* denotes the complex conjugate.) Also there is a complete orthonormal set of functions on $B$ which span $L^2(B)$ and for which there exists a Riesz-Fischer theory with convergence in $L^2$ norm. It follows immediately (Walsh [3, pp. 149–151]; Bergman [1, pp. 5–10]) that (1.1) may be applied to functions $f(z)$ in $L^2(B)$, and the result is the projection $f_1(z)$ of $f(z)$ on $L^2(B)$. That is,

$$
(1.1) \quad f_1(z) = \int_B \int_B d\xi d\eta f(\xi) K(z, \xi^*) \quad (\text{for } z \text{ in } B).
$$

Recently, Bergman and Schiffer [2] have introduced to the theory of kernel functions the new kernel

$$
L(z, \xi) = -\frac{2}{\pi} \frac{\partial^2 g(z, \xi)}{\partial z \partial \xi}.
$$

$g(z, \xi)$ is the Green's function for $B$, and

$$
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).
$$

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It is shown that

\[ L(z, \zeta) = \frac{1}{\pi(z - \zeta)^2} - l(z, \zeta), \]

where \( l(z, \zeta) \) is a single-valued analytic function in both variables on the closure of \( B \). In addition, if \( f_1(z) \) is in \( \mathcal{L}^2(B) \) and if

\[ \int_B \int_B d\xi d\eta L(z, \zeta) [f_1(\zeta)]^* \lim_{\epsilon \to 0} \int_B \int_{|\zeta - \xi| > \epsilon} d\xi d\eta L(z, \zeta) [f_1(\zeta)]^*, \]

then Bergman and Schiffer have proved that

\[ (1.2) \quad P \int_B \int_B d\xi d\eta L(z, \zeta) [f_1(\zeta)]^* = 0. \]

Next let \( f_3(z) \) be determined by the relation

\[ (1.3) \quad f(z) = f_1(z) + f_2(z). \]

Then \( f_3(z) \) belongs to the subspace which is the orthogonal complement to \( \mathcal{L}^2(B) \). Indeed,

\[ \int_B \int_B d\xi d\eta f_3(\zeta) [f_3(\zeta)]^* = 0. \]

It is the purpose of this paper to study the relationship of the \( L \)-kernel to \( \mathcal{L}^2(B) \). Just as the \( K \)-kernel determines the projection \( f_1(z) \) of \( f(z) \) on \( \mathcal{L}^2(B) \), so the \( L \)-kernel determines the projection \( f_2(z) \) of \( f(z) \) on the orthogonal complement to \( \mathcal{L}^2(B) \). In fact, if we define \( f(\bar{z}) = 0 \) exterior to \( B \) and take

\[ L(f) = \text{l.i.m.} \int_B \int_B d\xi d\eta L(z, \zeta) [f(\zeta)]^*, \]

then \( L[L(f)] = f_2(z) \) almost everywhere in \( B \). In addition, it is found that the subspace of functions which satisfy a Lipschitz condition of order \( \alpha, 0 < \alpha < 1 \), on \( B \) is invariant under the \( L \)-transformation.

2. A theorem of Beurling. In private conversation with this author, Beurling has communicated the following result.

**Theorem.** If (a) \( f(z) \in \mathcal{L}^2 \) (over the complex plane),

(b) \[ g_\alpha(z) = \frac{1}{\pi} \int_{|\zeta - z| > \epsilon} d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2}, \]

then there exists a function \( g(z) \) in \( \mathcal{L}^2 \) such that
(a) \[ g(z) = T(f) = \text{I.m.} \frac{1}{\pi} \int \int \frac{f(\xi)}{(|\xi - z|^2)^{1/2}} d\xi d\eta, \]

(b) \[ \{ T[T(f)]^* \}^* = T[T(f^*)]^* = f(z) \text{ a.e.}, \]

(c) \[ \int \int dxdy |f(z)|^2 = \int \int dxdy |g(z)|^2. \]

Unless otherwise indicated, the range of integration here and in what follows will be the entire complex plane. To prove the theorem, one first shows the existence of the Fourier-Plancherel transform of \( g(z) \). Indeed, the two-dimensional transform of \( g(z) \) is \(( -z^2/2\pi |z| ) F(x, y) \) where \((1/2\pi) F(x, y) \) is the transform of \( f(\xi) \). Application of this to the inversion formulae in the statement of the theorem yields the proof.

3. \( L(z, \xi) \) and the class \( L^2(B) \). Let \( f(z) \) be any function of \( L^2(B) \). We define \( f(z) = 0 \) exterior to \( B \). It is evident from the Schwarz inequality that \( \int_B d\xi d\eta L(z, \xi) [f(\xi)]^* \) converges absolutely. From this and Beurling’s theorem, one now finds that a function \( g_2(z) \) of summable square on \( B \) exists such that

\[ (3.1) \lim_{\epsilon \to 0} \left\| g_2(z) - \frac{1}{\pi} \int \int_{B: |z - \xi| > \epsilon} d\xi d\eta L(z, \xi) [f(\xi)]^* \right\|_B = 0. \]

Now \( f_2(z) \), as defined by (1.3), is orthogonal to each member of \( L^2(B) \). Therefore we may conclude that

\[ (3.2) \int \int_B d\xi d\eta L(z, \xi) [f_2(\xi)]^* = 0 \]

for each \( z \) in \( B \). It now follows from (1.2), (3.1) and (3.2) that

\[ (3.3) \lim_{\epsilon \to 0} \left\| g_2(z) - \frac{1}{\pi} \int \int_{B: |z - \xi| > \epsilon} d\xi d\eta [f_2(\xi)]^* \right\|_B = 0. \]

Accordingly, it is consistent to define \( g_2(z) \) in the entire complex plane by means of (3.3), for it reduces to (3.1) when \( z \) is in \( B \). When \( z \) is exterior to \( B \), \( g_2(z) \) vanishes since the kernel becomes an analytic function of \( \xi \) on \( B \). Next we prove a

**Lemma.** If \( z \) is in \( B \), then

\[ \int \int_B d\xi d\eta K(z, \xi^*) g_2(\xi) = 0. \]

For each positive \( \epsilon \), \((w = u + iv), \)
is of summable square on $B$, as is $K(z, \xi^*)$ with respect to $\xi$ for each fixed $z$ in $B$. We replace $g_2(z)$ in the lemma by its value (3.3). Then it follows from a well known theorem for sequences of functions convergent in the mean that

$$
\lim_{t \to 0} \iint_{B:|t-w|>\epsilon} d\xi d\eta K(z, \xi^*) \frac{1}{\pi} \iint_{B:|t-w|>\epsilon} d\xi d\eta \frac{[f_2(w)]^*}{(t-w)^2} = \lim_{t \to 0} \iint_{B} d\xi d\eta K(z, \xi^*) \frac{1}{\pi} \iint_{B:|t-w|>\epsilon} d\xi d\eta \frac{[f_2(w)]^*}{(t-w)^2}.
$$

The integrals on the right are absolutely convergent for each $z$ in $B$, so we may interexchange the order of integration. This produces

$$
\lim_{t \to 0} \iint_{B} d\xi d\eta [f_2(w)]^* \frac{1}{\pi} \iint_{B:|t-w|>\epsilon} d\xi d\eta \frac{K(z, \xi^*)}{(w-\xi)^2} = \iint_{B} d\xi d\eta [f_2(w)]^* \frac{1}{\pi} \iint_{B:|t-w|>\epsilon} d\xi d\eta \frac{K(z, \xi^*)}{(w-\xi)^2}.
$$

It is a result of Bergman and Schiffer [2, p. 216] that

$$
l(z, w) = \frac{1}{\pi} P \iint_{B} d\xi d\eta [K(\xi, w^*)]^* \frac{1}{(\xi-z)^2},
$$

and the existence of this principal value implies that it is the same as the result obtained by taking the l.i.m. Inasmuch as $[K(\xi, w^*)]^* = K(w, \xi^*)$, it now follows that the right member of (3.4) is

$$
\iint_{B} d\xi d\eta [f_2(w)]^* l(z, w).
$$

Since this is zero, the lemma is proved.

We may now state

**Theorem 3.1.** If $f(z) \in L^2(B)$, then

$$L[L(f)] = f_2(z) \text{ a.e. in } B.$$

If we define $f_2(z) = 0$ for $z$ exterior to $B$, then we may write $g_2(z) = T(f_2^*)$. It follows from Beurling's theorem that $T(g_2^*) = f_2(z)$. Since $g_2(z) = 0$ for $z$ exterior to $B$, we have

$$T(g_2^*) = \lim_{t \to 0} \frac{1}{\pi} \iint_{B:|t-z|>\epsilon} d\xi d\eta \frac{[g_2(\xi)]^*}{(z-\xi)^2}.$$
But the lemma implies that \( g_2(z) \) belongs to the orthogonal complement to the space \( L^a(B) \). Therefore

\[
L(g_2) = -\int_B d\xi d\eta (z, \xi) [g_2(\xi)* + 1.i.m. \int_{\epsilon \to 0} \int_{B:|z-\xi|>|z|} d\xi d\eta \frac{[g_2(\xi)]*}{(z-\xi)^2}
\]

\[
= T(g_2*) = f_2(z).
\]

Since \( L(g_2) = L[L(f)] \), the theorem follows.

**Corollary.** \( \|f_2\|_B = \|g_2\|_B \).

**Corollary.** If \( f(z) \in L^a(B) \), then \( L[L(f)] = 0 \) a.e. in \( B \) if and only if \( f(z) \in L^a(B) \).

**4. Invariance of the class of functions Lip \( \alpha \) on \( B \).** We first consider the class of functions which satisfy a Lipschitz condition uniformly over the complex plane.

**Theorem 4.1.** If (a) \( \int dxdy |f(z)|^2 < \infty \),
(b) \( |f(z_1) - f(z_2)| \leq A |z_1 - z_2|^{\alpha a} \) when \( |z_1 - z_2| < \delta \),
(c) \( 0 < \alpha < 1 \),

then \( g(z) = T(f) \) satisfies a Lipschitz condition of order \( \alpha \).

Under the conditions stated, the principal value integral

\[
\frac{1}{\pi} P \int d\xi d\eta \frac{f(\xi)}{(\xi - z)^2}
\]

exists. Inasmuch as

\[
\frac{1}{\pi} P \int_{|\xi - z| < \delta} d\xi d\eta \frac{1}{(\xi - z)^2} = 0,
\]

it follows that

\[
\frac{1}{\pi} P \int d\xi d\eta \frac{f(\xi)}{(\xi - z)^2} = \frac{1}{\pi} \int_{|\xi - z| > \delta} d\xi d\eta \frac{f(\xi)}{(\xi - z)^2} + \frac{1}{\pi} \int_{|\xi - z| < \delta} d\xi d\eta \frac{f(\xi) - f(z)}{(\xi - z)^2}.
\]

We note that the last integral on the right is absolutely convergent because of the Lipschitz condition on \( f(z) \). The first integral on the right also converges, and thus the principal value integral exists.

It is also evident that the conditions of the theorem imply that \( f(z) \) is continuous and bounded in the \( z \)-plane, tending uniformly to zero as \( |z| \to \infty \).
Now consider any two points $z_1$ and $z_2$ such that $|z_1 - z_2| < \delta$. A short calculation yields

$$g(z_1) - g(z_2) = \frac{2(z_1 - z_2)}{\pi} P. \int \int d\xi d\eta \left[ f(\xi) - f(z_1) \right] \frac{\xi - (z_1 + z_2)/2}{(\xi - z_1)^2(\xi - z_2)^2}.$$  \hfill (4.1)

Here $P.$ indicates that the principal value is to be calculated with respect to both singularities of the integrand. We observe that the region $C_1$: $|z_2 - z| \leq 2\delta$ contains $C_{12}$: $|z_2 - z| \leq 2|z_1 - z_2|$, and we denote by $M$ and $N$ the regions given respectively by $M$: $\{|z_2 - z| \leq 2\delta, |z_2 - z| > 2|z_1 - z_2| \}$ and $N$: $\{|z_2 - z| \leq 2|z_1 - z_2| \}$. If $\xi$ belongs to $M$, then it may be verified from the geometry that

$$\frac{1}{2} \leq \left| \frac{\xi - z_2}{\xi - z_1} \right| \leq 2, \quad \frac{1}{2} \leq \left| \frac{\xi - (z_1 + z_2)/2}{\xi - z_2} \right| \leq 2.$$  \hfill (4.2)

Designate by $I$, $II$, and $III$ respectively those parts of the right side of (4.1) taken over the regions $M$, $N$, and $C_1$: $|z_2 - z| > 2\delta$. Using the inequalities (4.2) and the Lipschitz condition for $f(z)$, the integral $I$ is in absolute value less than or equal to

$$\frac{2^{4+\alpha}A}{\pi} \int \int_M d\xi d\eta \left| \xi - z_2 \right| \leq \frac{2^{4+\alpha}A}{1 - \alpha} \left[ \left| z_1 - z_2 \right|^\alpha - \delta^{\alpha-1} \left| z_1 - z_2 \right| \right] = O\left( \left| z_1 - z_2 \right|^\alpha \right).$$

We rewrite the integral $II$ in the form

$$\frac{1}{\pi} P. \int \int_N d\xi d\eta \left[ f(\xi) - f(z_1) - f(z_2) + f(z_2) - f(z_1) \right].$$

The third of these integrals is zero, and the first two converge absolutely in accordance with the Lipschitz condition for $f(z)$. An estimate for each of the first two integrals is readily obtained. We have

$$\left| \int \int_N d\xi d\eta \frac{f(\xi) - f(z_1)}{(\xi - z_1)^2} \right| \leq A \int \int_{|z_1| \leq 2|z_1 - z_2|} d\xi d\eta \left| \xi - z_1 \right| \leq \frac{2\pi A}{\alpha} 3^\alpha \left| z_1 - z_2 \right|^\alpha = O\left( \left| z_1 - z_2 \right|^\alpha \right).$$
The estimate is uniform for all $z_1$ and $z_2$ for which $|z_1 - z_2| < \delta$. The same estimate may be obtained for the second integral.

If $\zeta$ is in $\mathbb{C}$, the inequalities (4.2) are still valid. Using these, we may estimate the integral III. Indeed, in absolute value, it is less than or equal to

$$\frac{16}{\pi} \left| z_1 - z_2 \right| \int_0^\infty \int_{\mathbb{C}} d\zeta d\eta \left| f(\zeta) \right| |\zeta - z_2|^2 \leq \frac{16}{\pi} \left| z_1 - z_2 \right| \left[ \int_0^\infty \int_{\mathbb{C}} d\zeta d\eta \left| f(\zeta) \right|^2 \int_0^\infty \int_{\mathbb{C}} d\zeta d\eta \frac{1}{|\zeta - z_2|^\alpha} \right]^{1/2}$$

uniformly for all $z_1$ and $z_2$ for which $|z_1 - z_2| < \delta$. Thus it follows that (4.1) is uniformly $O(|z_1 - z_2|^\alpha)$, and the theorem is proved.

It is important to note that the Lipschitz condition on $f(z)$ need not hold uniformly. For example, if $f(z)$ satisfies a Lipschitz condition of order $\alpha$ at $z_0$, then $g(z)$ satisfies the same order Lipschitz condition at $z_0$. With this remark, we have immediately:

**Theorem 4.2.** Let $f(z)$ belong to $L^2(B)$ and satisfy a Lipschitz condition of order $\alpha$ at a point $z_0$ in $B$. Then $L(f)$ satisfies the same order Lipschitz condition at $z_0$. Moreover, if $f(z)$ satisfies a Lipschitz condition of order $\alpha$ uniformly on $B$, so does $L(f)$.

It follows from the previous remarks that the part of $L(f)$ which results from the singular kernel satisfies a Lipschitz condition of order $\alpha$. The part which comes from $l(z, \zeta)$ satisfies a Lipschitz condition of order 1 because of the analyticity of $l(z, \zeta)$.

5. $L(z, \zeta)$ and the unit circle. For the unit circle,

$$L(z, \zeta) = \frac{1}{\pi(1 - \zeta z)^2}, \quad K(z, \zeta^*) = \frac{1}{\pi(1 - \zeta^* z)^2}.$$  

While the general results already obtained easily reduce to the special case of the unit circle, the simplicity of $L(z, \zeta)$ leads to special results which may be of some interest.

Let $f(z)$ be a complex-valued function on $B$: $|z| < 1$ such that

$$\int_0^\infty \int_{|z| < 1} dx dy |f(z)|^2 < \infty.$$  

Define $f(z)$ and its two components $f_1(z)$ and $f_2(z)$, as given by (1.3), to be zero for $|z| > 1$. We consider the transformation
When \(|z| < 1\), it follows immediately from the classical kernel theory that \(K(f) = f_1(z)\). On the other hand, if \(|z| > 1\), we have \(K(f) = K(f_1) + K(f_2)\). Application of Green's theorem with respect to the region \(B_2\): \(\epsilon < |\frac{1}{z} - 1/z^*| < 1\) shows that \(K(f_1) = 0\), and from (3.3), we find that \(K(f_2) = 1/z^2[g_2(1/z^*)]^*\). Then we have

\[
K(f) = \frac{1}{z^2}\left[ g_2\left(\frac{1}{z^*}\right) \right]^* \quad \text{for } |z| > 1.
\]

To compute \(K^2(f)\), we first observe that \(g_2(1/z) = 0\) for \(|z| < 1\), and \(f_1(z) = 0\) for \(|z| > 1\). Then

\[
K^2(f) = f_1(z) + K\left[ \frac{1}{z^2}\left[ g_2\left(\frac{1}{z^*}\right) \right]^* \right] \quad \text{a.e.}
\]

The last member on the right side is

\[
\lim_{\epsilon \to 0} \frac{1}{\pi} \int \int_{|\xi - 1/z^*| > \epsilon} d\xi d\eta \frac{1}{(1 - \xi^* z)^2} g_2(1/z^*)^* \left[ g_2\left(\frac{1}{z^*}\right) \right]^*
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{\pi} \int \int_{|w-z| > \epsilon} dw dv \frac{g_2(w)^*}{(w - z)^2}
\]

since the Jacobian of the transformation \(\xi = 1/w^*\) is \(1/|w|^4\). It can be shown that this last integral is equal to

\[
\lim_{\epsilon \to 0} \frac{1}{\pi} \int \int_{|w-z| > \epsilon} dw dv \frac{g_2(w)^*}{(w - z)^2},
\]

whence it follows that \(K^2(f) = f_1(x) + [T^*(g_2)]^* = f_1(x) + f_2(x)\) a.e. Then our result is that

\[
K^2(f) = f(x) \quad \text{a.e.}
\]

**BIBLIOGRAPHY**


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