ON THE CONVERGENCE-ABSCISSAS OF THE GENERALIZED FACTORIAL SERIES

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1. Introduction. We consider the generalized factorial series

\[
F(s) = \sum_{n=1}^{\infty} a_n \left[ \lambda_1 \lambda_2 \cdots \lambda_n \right] \left[ (s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n) \right]^{-1},
\]

\[ s = \sigma + it, \quad \lambda_n = r_n e^{i\phi_n} (n = 1, 2, \cdots), \]

where

\[
\lim_{n \to \infty} r_n = +\infty, \quad |\phi_n| \leq \phi < \pi/2 \quad (n = 1, 2, \cdots).
\]

In his classical note [1, §6], E. Landau has studied (1.1) in the case in which \( \sum_{n=1}^{\infty} 1/r_n = +\infty, \quad \phi_n = 0 \) \((n = 1, 2, \cdots)\). Under additional conditions, he has determined convergence-abscissas of (1.1) in terms of coefficients \( a_n \) \((n = 1, 2, \cdots)\). S. Pincherle [2], G. Belardinelli [3], and T. Fort [4, 5] have studied (1.1) with complex \( \lambda_n \) \((n = 1, 2, \cdots)\) satisfying (1.2) and some other conditions. In this note, without any additional conditions, we shall determine the convergence-abscissas of (1.1) with real \( \lambda_n \) \((n = 1, 2, \cdots)\) in terms of coefficients \( a_n \) \((n = 1, 2, \cdots)\). In the case in which the \( \lambda_n \) are complex, the convergence-domains of (1.1) are not generally half-planes, and so the convergence-abscissas of (1.1) have no meaning.

The main theorems are:

**Theorem I.** In the case \( \phi_n = 0 \) \((n = 1, 2, \cdots)\), (1.1) has three convergence-abscissas, i.e. a simple convergence-abscissa \( \sigma_s \), a uniform convergence-abscissa \( \sigma_u \), and an absolute convergence-abscissa \( \sigma_A \) such that \( \sigma_s = \sigma_u \leq \sigma_A \).

**Remark.** (1) In the convergence-problem of (1.1), the sequence of points \(-\lambda_n \) \((n = 1, 2, \cdots)\) is excluded from the \( s \)-plane by small circles with centres at \(-\lambda_n \) \((n = 1, 2, \cdots)\) and radii \( \epsilon, \epsilon \) being a small positive constant.

(2) The divergence of \( \sum_{n=1}^{\infty} 1/r_n \) is not necessary for the validity of Theorem 1.

**Theorem II.** If \( \sum_{n=1}^{\infty} 1/r_n < +\infty \), the necessary and sufficient condition for (1.1) to be simply (absolutely) convergent at \( s = s_0 \) distinct from \(-\lambda_n \) \((n = 1, 2, \cdots)\) is that \( \sum_{n=1}^{\infty} a_n \left( \sum_{n=1}^{\infty} |a_n| \right) \) converges. If

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furthermore $\phi_n = 0$ ($n = 1, 2, \cdots$), then three possibilities now present themselves:

| Case | $\sum_{n=1}^{\infty} |a_n|$ | $\sum_{n=1}^{\infty} a_n$ | $\sigma_a = \sigma_u$ | $\sigma_a$ |
|------|----------------|------------------|----------------|------|
| I    | $< +\infty$   | convergent       | $= -\infty$   | $= -\infty$ |
| II   | $= +\infty$   | convergent       | $= -\infty$   | $= +\infty$ |
| III  | $= +\infty$   | divergent        | $= +\infty$   | $= +\infty$ |

**THEOREM III.** If $\sum_{n=1}^{\infty} 1/r_n = +\infty$, $\phi_n = 0$ ($n = 1, 2, \cdots$), then the three convergence-abscissas of (1.1) are determined respectively by

(a) $\sigma_u = \limsup \frac{1}{\log n} \left( \sum_{n=1}^{\infty} a_n \exp (\phi(l_n) - \phi(l_n)) \right)$,

(b) $\sigma_a = \limsup \frac{1}{\log n} \left\{ \sum_{n=1}^{\infty} |a_n| \exp (\phi(l_n) - \phi(l_n)) \right\}$,

where

(c) $l_n = \sum_{i=1}^{n} l/r_i$, $(0 < l_1 < l_2 < \cdots < l_n \to +\infty)$,

(d) $\phi(x)$ is the positive and differentiable function defined for $x > 0$ such that

(i) $\phi(x) \uparrow +\infty$, $\phi'(x) \to +\infty$ as $x \to +\infty$.

(ii) for any given $\epsilon > 0$, $\int_{x}^{+\infty} \exp (-\epsilon x) |\phi'(x)| \, dx < +\infty$.

**COROLLARY I (Equiconvergence Theorem)** (T. Fort [4, p. 239]). Under the same conditions as in Theorem III, (1.1) has the same abscissa of simple convergence and the same abscissa of absolute convergence as the Dirichlet series

$$G(s) = \sum_{n=1}^{\infty} a_n \exp (-l_n s).$$

**COROLLARY II.** Under the same conditions as in Theorem III, we have

(a) $\sigma_a = \sigma_u = \limsup \frac{1}{x} \log \left( \sum_{|x| \leq l_n < x} a_n \right)$,

(b) $\sigma_a = \limsup \frac{1}{x} \log \left\{ \sum_{|x| \leq l_n < x} |a_n| \right\}$,
where \( [x] \) denotes the greatest integer contained in \( x \).

(b) \( 0 \leq \sigma_n - \sigma_n \leq \limsup_{n \to \infty} \frac{1}{\ln n} \log n. \)

2. Proof of Theorem I. We first prove some necessary lemmas, which are analogues of theorems concerning ordinary factorial series [6, pp. 171–174].

**Lemma I.** If (1.1) is simply convergent at \( s = s_0 \), then (1.1) is uniformly convergent in the angular domain \( D(s_0, \vartheta, \phi) : |\arg (s - s_0)| \leq \theta < (\pi/2 - \phi) \), where \( \theta \) is an arbitrary but fixed constant.

As a special case of Lemma I, we have

**Lemma I'.** If (1.1) with real \( \lambda_n (n = 1, 2, \cdots) \) is simply convergent at \( s = s_0 \), then (1.1) is uniformly convergent in the angular domain \( D(s_0, \vartheta, 0) : |\arg (s - s_0)| \leq \theta < \pi/2 \), where \( \theta \) is an arbitrary but fixed constant.

Under the assumptions that \( \lim_{n \to \infty} \phi_n = 0 \), and \( \sum_{n=1}^{\infty} 1/r_n = +\infty \), T. Fort [4, p. 237, Theorem IV] has proved that (1.1) converges uniformly in the angular domain \( D(s_0, \vartheta, 0) \), provided that it converges simply at \( s = s_0 \). Since we can put \( \phi = \epsilon \) in Lemma I, \( \epsilon \) being any small positive constant, this theorem is evidently contained in Lemma I.

**Proof of Lemma I.** We first establish the inequality

\[(2.1) \quad |s + \lambda_n| > |s_0 + \lambda_n| + r \sin (\eta/2) \quad \text{for } n \geq n_1,\]

where

(i) \( s \in D(s_0, \vartheta, \phi) \), \( r = |s - s_0| \), \( \vartheta = \pi/2 - (\phi + \eta) \) (\( \eta > 0 \)),

(ii) \( n_1 \) is a sufficiently large integer.

In fact, putting \( \theta = \arg (s - s_0) - \arg (s_0 + \lambda_n) \), where \( s \in D(s_0, \vartheta, \phi) \), we have easily

\[\frac{\pi}{2} + \frac{\eta}{2} \leq \theta < \frac{3\pi}{2} - \frac{\eta}{2} \quad \text{for } n \geq n_1,\]

so that

\[|s + \lambda_n|^2 = r^2 + |s_0 + \lambda_n|^2 - 2r |s_0 + \lambda_n| \cos \theta \]
\[\geq \{ |s_0 + \lambda_n| + r \sin (\eta/2) \}^2 \quad \text{for } n \geq n_1,\]

which proves (2.1). Let us put

\[b_n = a_n [\lambda_1 \cdots \lambda_n] [(s_0 + \lambda_1)(s_0 + \lambda_2) \cdots (s_0 + \lambda_n)]^{-1},\]

\[c_n(s) = [(s_0 + \lambda_1)(s_0 + \lambda_2) \cdots (s_0 + \lambda_n)] [(s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n)]^{-1}.\]
Equation (2.1) yields
\[
\begin{align*}
| (s_0 + \lambda_n) / (s + \lambda_n) | &< \rho_n \left[ \rho_n + r \sin \left( \eta/2 \right) \right]^{-1}, \quad \text{for } n \geq n_1, \\
| (s - s_0) / (s + \lambda_{n+1}) | &< r \left[ \rho_{n+1} + r \sin \left( \eta/2 \right) \right]^{-1},
\end{align*}
\]
where \( s \in \mathcal{D}(s_0, \theta, \phi) \), \( r = | s - s_0 | \), and \( \rho_n = | s_0 + \lambda_n | \). Hence
\[
| c_n(s) - c_{n+1}(s) | = | c_n(s)(s - s_0)(s + \lambda_{n+1})^{-1} | < \left| K(s) \right| \cdot d_n \cdot r \left[ \rho_{n+1} + r \sin \left( \eta/2 \right) \right]^{-1},
\]
where
\[
K(s) = \left[ (s_0 + \lambda_1)(s_0 + \lambda_2) \cdots (s_0 + \lambda_{n-1}) \right] \left[ (s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_{n-1}) \right]^{-1},
\]
and
\[
d_n = \prod_{i=n_1}^{n} \rho_n \left[ \rho_n + r \sin \left( \eta/2 \right) \right]^{-1}.
\]

In \( \mathcal{D}_0 \), which we get by excluding from \( \mathcal{D}(s_0, \theta, \phi) \) the sequence of circles with centres at \( -\lambda_n \) (\( n = 1, 2, \cdots \)) and radii \( \epsilon \), \( \epsilon \) being a small positive constant, we have evidently
\[
| K(s) | < K,
\]
where \( K \) is a suitable constant. Since
\[
d_n \cdot r \cdot \left[ \rho_{n+1} + r \sin \left( \eta/2 \right) \right]^{-1} = \cosec \left( \eta/2 \right) (d_n - d_{n+1}),
\]
taking account of (2.4) and (2.5), we have for any large \( N \)
\[
\begin{align*}
\sum_{n=n_1}^{N} | c_n(s) - c_{n+1}(s) | &< K \cosec \left( \eta/2 \right) \sum_{n=n_1}^{N} (d_n - d_{n+1}) \\
&< K \cosec \left( \eta/2 \right) d_{n_1}
\end{align*}
\]
uniformly in \( \mathcal{D}_0 \).

Since \( \sum_{n=1}^{\infty} b_n \) is convergent by the hypothesis, on account of (2.6) and du Bois-Reymond's Theorem [7, p. 315], \( F(s) = \sum_{n=1}^{\infty} b_n c_n(s) \) is uniformly convergent in \( \mathcal{D}_0 \). q.e.d.

**Lemma II.** If (1.1) is absolutely convergent at \( s = s_0 \), then
\[
\sum_{n=1}^{\infty} | a_n | \left| (\lambda_1 \lambda_2 \cdots \lambda_n) [(s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n)]^{-1} \right| \text{ is uniformly convergent in the angular domain } \mathcal{D}(s_0, \theta, \phi), \text{ where } \mathcal{D}(s_0, \theta, \phi) \text{ has the same meaning as in Lemma I.}
\]

As a corollary, we get

**Lemma II'.** If (1.1) with real \( \lambda_n \) (\( n = 1, 2, \cdots \)) is absolutely convergent at \( s = s_0 \), then (1.1) is absolutely and uniformly convergent in the
Proof of Lemma II. Using the same notation as in Lemma I, (2.1) and (2.3) are also valid. Since
\[ \left| c_n(s) - c_{n+1}(s) \right| = \left| c_n(s) \right| \cdot \left| 1 - \left( s_0 + \lambda_{n+1}\right)(s + \lambda_{n+1})^{-1} \right| \leq \left| c_n(s) \right| \cdot \left| (s - s_0)(s + \lambda_{n+1})^{-1} \right|, \]
on account of (2.4) and (2.5), we obtain for any large \( N \)
\[ \sum_{n=1}^{N} \left| c_n(s) - c_{n+1}(s) \right| < K \csc \left( \pi/2 \right) \cdot d_n \]
uniformly in \( D_0 \). Since \( \sum_{n=1}^{\infty} |b_n| \) is convergent by the hypothesis, it results by virtue of (2.7) and du Bois-Reymond's theorem that \( \sum_{n=1}^{\infty} |b_n \cdot c_n(s)| \) is uniformly convergent in \( D_0 \). q.e.d.

Lemma III. If (1.1) is simply convergent at \( s = s_0 \), and furthermore there exists a point \( s_1 \) contained in the angular domain \( D(s_0, \pi/2 - \phi) \): \( |\arg (s - s_0)| \leq \pi/2 - \phi \), such that for a sufficiently large integer \( n_1 \), we have
\[ |\arg (s_1 + \lambda_n)| \leq \phi \]
for \( n \geq n_1 \), then (1.1) is uniformly convergent in the angular domain \( D(s_2, \pi/2 - \phi) \), where \( s_2 = s_1 + \varepsilon \sec \phi \), \( \varepsilon \) being any small positive constant.

As an immediate consequence of Lemma III, we get

Lemma III'. If (1.1) with real \( \lambda_n \) (\( n = 1, 2, \cdots \)) is simply convergent at \( s = s_0 \), then (1.1) is uniformly convergent in the half-plane \( D: \Re(s) \geq \Re(s_0) + \varepsilon \), \( \varepsilon \) being any small positive constant.

In fact, we can put \( \phi = 0, s_1 = \Re(s_0) \), and \( s_2 = \Re(s_0) + \varepsilon \) in Lemma III.

Proof of Lemma III. We first prove
\[ |s + \lambda_n| \geq |s_2 + \lambda_n| + \varepsilon/2 \]
for \( n \geq n_1 \), where \( s \in D(s_2, \pi/2 - \phi) \), and \( s_2 = s_1 + \varepsilon/2 \cdot \sec \phi \). In fact, putting \( \alpha_n = \arg (s_2 + \lambda_n) \), we have evidently
\[ |\alpha_n| \leq \phi \]
for \( n \geq n_1 \).

Projecting the vector \( s + \lambda_n \) perpendicularly on the vector \( s_2 + \lambda_n \), we get easily
\[ |s + \lambda_n| \geq |s_2 + \lambda_n| + \varepsilon/2 \cdot \sec \phi \cdot \cos \alpha_n, \]
so that, by (2.9),
\[ |s + \lambda_n| \geq |s_2 + \lambda_n| + \varepsilon/2, \]
which proves (2.8).

Let us put

\[ b_n = a_n \left[ \lambda_1 \cdots \lambda_n \right]\left( (s_3 + \lambda_1)(s_3 + \lambda_2) \cdots (s_3 + \lambda_n) \right)^{-1}, \]

(2.10) \[ c_n(s) = \left( (s_3 + \lambda_1)(s_3 + \lambda_2) \cdots (s_3 + \lambda_n) \right) \cdot \left( (s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n) \right)^{-1}. \]

By (2.8) and arguments similar to those employed in the proof of Lemma I, we have

\[ | c_n(s) - c_{n+1}(s) | = | c_n(s) | \cdot | (s - s_3)(s + \lambda_{n+1})^{-1} | \]

\[ < | K(s) | \cdot d_n \cdot (\rho_n + \epsilon/2)^{-1}, \]

where

\[ K(s) = (s - s_3) \left[ (s_3 + \lambda_1) \cdots (s_3 + \lambda_{n-1}) \right] \cdot \left( (s + \lambda_1) \cdots (s + \lambda_{n-1}) \right)^{-1}, \]

\[ \rho_n = | s_3 + \lambda_n |, \quad d_n = \prod_{i=1}^{n} \rho_i (\rho_i + \epsilon/2)^{-1}. \]

Since \( d_n (\rho_n + \epsilon/2)^{-1} = 2/\epsilon \cdot (d_n - d_{n+1}) \), and \( K(s) = O(1) \) in the domain \( D_0 \), as is easily seen by excluding from \( D(s_3, \pi/2 - \phi) \) the sequence of small circles with centres at \( -\lambda_n \) (\( n = 1, 2, \cdots \)) and radii \( \epsilon' > 0 \), by virtue of (2.11) we have

\[ | c_n(s) - c_{n+1}(s) | < 2K/\epsilon \cdot (d_n - d_{n+1}) \]

uniformly in \( D_0 \), where \( K \) is a suitable constant. Hence

(2.12) \[ \sum_{n=1}^{N} \left| c_n(s) - c_{n+1}(s) \right| < 2K/\epsilon \cdot (d_n - d_{N+1}) < 2K/\epsilon \cdot d_n \]

uniformly in \( D_0 \) for any given \( N \).

Since (1.1) is simply convergent at \( s = s_0 \) by virtue of Lemma I, it follows from (2.12) and du Bois-Reymond’s theorem that \( F(s) = \sum_{n=1}^{\infty} b_n c_n(s) \) is uniformly convergent in \( D_0 \). q.e.d.

Now we are in a position to prove Theorem I.

**Proof of Theorem I.** If (1.1) is simply (absolutely) convergent at \( s = s_0 \), then (1.1) is also simply (absolutely) convergent at \( s = s_1 \) with \( \Re(s_0) < \Re(s_1) \) by virtue of Lemma I’ (Lemma II’’). Hence there exists a simple (absolute) convergence-abscissa \( \sigma_a(\sigma_a) \) of (1.1), and we have evidently \( \sigma_a \leq \sigma_a \).

For any given \( \epsilon > 0 \), (1.1) is simply convergent at \( s = \sigma_a + \epsilon/2 \), so that by Lemma III’', (1.1) is uniformly convergent for \( \Re(s) \geq \sigma_a + \epsilon \).

But since (1.1) is not simply convergent on \( s = \sigma_a - \epsilon \), (1.1) is not uni-
formly convergent for $\Re(s) \geq \sigma_s - \varepsilon$. Hence $\sigma_u$ coincides with $\sigma_s$. Thus we have $\sigma_u = \sigma_v = \sigma_s$. q.e.d.

3. Proof of Theorem II. Since $\sum_{n=1}^{\infty} 1/r_n < +\infty$, the infinite product $g(s) = \prod_{n=1}^{\infty} (1 + s/\lambda_n)$ converges, so that we have

\[ (3.1) \quad 0 < |g(s)| < +\infty \quad \text{for} \quad s \neq -\lambda_n \quad (n = 1, 2, \cdots). \]

Let us put

\[ c_n(s) = \left[ \lambda_1 \cdots \lambda_n \right] [(s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n)]^{-1} = [g_n(s)]^{-1}, \]

where $g_n(s) = \prod_{n=1}^{n} (1 + s/\lambda_i)$. Since

\[ c_n(s) - c_{n+1}(s) = [g_n(s) \cdot \lambda_{n+1}]^{-1} \cdot s(1 + s/\lambda_{n+1})^{-1}, \]

by (3.1) we get

\[ |c_n(s) - c_{n+1}(s)| < K_1 |g(s)|^{-1} \cdot 1/r_{n+1} \quad \text{for} \quad n \geq n_1, \]

where (i) $K_1$ is a suitable constant, (ii) $n_1$ is a sufficiently large integer. Hence

\[ (3.2) \quad \sum_{n=n_1}^{\infty} |c_n(s) - c_{n+1}(s)| < K_1 |g(s)|^{-1} \cdot \sum_{n=n_1}^{\infty} 1/r_{n+1} < +\infty. \]

If $\sum_{n=1}^{\infty} a_n$ converges, then by (3.2) and du Bois-Reymond’s theorem, $F(s) = \sum_{n=1}^{\infty} a_n c_n(s)$ also converges for $s$ different from $-\lambda_n$ ($n = 1, 2, \cdots$).

Next suppose that $F(s_0) = \sum_{n=1}^{\infty} b_n(s_0)$ converges for $s = s_0 \neq -\lambda_n$ ($n = 1, 2, \cdots$), where

\[ b_n(s_0) = a_n [\lambda_1 \cdots \lambda_n] [(s_0 + \lambda_1)(s_0 + \lambda_2) \cdots (s_0 + \lambda_n)]^{-1}. \]

Since $g_{n+1}(s_0) - g_n(s_0) = g_n(s_0) \cdot s_0/\lambda_{n+1}$, by (3.1) we get

\[ |g_{n+1}(s_0) - g_n(s_0)| < |g(s_0)| \cdot K_2/r_{n+1} \quad \text{for} \quad n \geq n_1, \]

where (i) $K_2$ is a suitable constant, (ii) $n_2$ is a sufficiently large integer, so that

\[ (3.3) \quad \sum_{n=n_2}^{\infty} |g_{n+1}(s_0) - g_n(s_0)| < |g(s_0)| \cdot K_2 \cdot \sum_{n=n_2}^{\infty} 1/r_{n+1} < +\infty. \]

Since $\sum_{n=1}^{\infty} b_n(s_0)$ converges, by (3.3) and du Bois-Reymond’s theorem, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n(s_0) g_n(s_0)$ is also convergent.

By entirely similar arguments, we can prove that the necessary-sufficient condition for (1.1) to converge absolutely at $s = s_0$ different from $-\lambda_n$ ($n = 1, 2, \cdots$) is that $\sum_{n=1}^{\infty} |a_n| < +\infty$.

If $\sum_{n=1}^{\infty} 1/r_n < +\infty$ and $\phi_n = 0$ ($n = 1, 2, \cdots$), then the second part
of Theorem II immediately follows from Theorem I and what is proved above.

4. Proof of Theorem III. Let us put

\[ k = \limsup_{n \to \infty} 1/\ln n \cdot \log \left| \sum_{i=1}^{n} a_i \exp(\phi(l_i) - \phi(l_n)) \right|. \]

We shall first establish the inequality

\[ k \leq \sigma. \]

Since (1.1) is simply convergent for \( s = \sigma > \sigma_n \), there exists a constant \( K \) such that

\[ |S_n| < K \quad (n = 1, 2, \ldots) \]

where

\[ S_n = \sum_{i=1}^{n} a_i \left[ \lambda_1 \cdots \lambda_i \right] \left[ (\sigma + \lambda_1)(\sigma + \lambda_2) \cdots (\sigma + \lambda_i) \right]^{-1}. \]

Putting \( S_0 = 0 \) and applying Abel's transformation, we have

\[ \sum_{i=1}^{n} a_i \exp(\phi(l_i)) = \sum_{i=1}^{n-1} S_i (f(i) - f(i+1)) + S_n f(n), \]

where \( f(i) = \exp(\phi(l_i)) \cdot \prod_{n=1}^{i} (1 + \sigma/\lambda_n) \). On the other hand,

\[ f(i) = Q(\sigma) \cdot \exp \left\{ \phi(l_i) + l_i (\sigma + \epsilon_i(\sigma)) \right\} \quad \text{for } i > n_1, \]

where

(i) \( Q(\sigma) = \prod_{n=1}^{n_1} (1 + \sigma/\lambda_n) \exp(-\sigma/\lambda_n), \)

(ii) \( \lim_{i \to \infty} \epsilon_i(\sigma) = 0, \)

(iii) \( n_1 \) is a sufficiently large integer.

In fact, since

\[ (1 + x) = \exp(x + x^2 \cdot \rho(x)), \quad |\rho(x)| \leq 1 \quad \text{for } |x| \leq 1/2, \]

we can easily obtain the relation

\[ f(i) = \prod_{n=1}^{n_1} (1 + \sigma/\lambda_n) \exp(-\sigma/\lambda_n) \]

\[ \times \exp \left\{ \phi(l_i) + \sigma l_i + \sigma^2 \cdot \vartheta(\sigma) \left( \sum_{n=1}^{i} 1/\lambda_n^2 \right) \right\}, \]
where (i) $|\sigma/\lambda_n| \leq 1/2$ for $n > n_1$, (ii) $|\theta(\sigma)| \leq 1$. Since $\lim_{t \to \infty} 1/t \cdot \sum_{n=1}^t 1/\lambda_n^2 = 0$, (4.6) gives (4.5).

Taking account of the hypothesis (d) part (i), we can easily prove that

$$g(i) \uparrow \infty$$

for $i > n_2$.

where

(i) $g(i) = \exp (\phi(l_i) + l_i(\sigma + \epsilon_i(\sigma)))$,

(ii) $n_2$ is a sufficiently large integer.

Therefore, putting $N = \text{Max} (n_1, n_2)$, by (4.4) and (4.3) we have

$$\left| \sum_{i=1}^n a_i \exp (\phi(l_i)) \right| \leq K \cdot \left| \sum_{i=1}^N f(i) - f(i + 1) \right| + K \cdot |Q(\sigma)| \cdot \left\{ \sum_{i=N+1}^{n-1} g(i + 1) - g(i) + g(n) \right\},$$

so that for sufficiently large $n$,

$$\left| \sum_{i=1}^n a_i \exp (\phi(l_i)) \right| < 3K \cdot |Q(\sigma)| \cdot g(n).$$

Hence $k \leq \sigma + \lim_{n \to \infty} e_n(\sigma) = \sigma$. Letting $\sigma \to \sigma*$, we have $k \leq \sigma*$, which proves (4.2).

Next we shall prove

$$(4.7) \quad \sigma* \leq k.$$  

By the definition of $k$, for any given $\delta > 0$, there exists a constant $N$ such that

$$(4.8) \quad |T_n| < U_n = \exp \{\phi(l_n) + l_n(\sigma + \delta/2)\} \quad \text{for} \quad n \geq N,$$

where $T_n = \sum_{i=1}^n a_i \exp (\phi(l_i))$. Taking account of $a_n = (T_n - T_{n-1}) \exp (-\phi(l_n))$, by Abel's transformation we get

$$\sum_{i=N+1}^M a_i \prod_{k=1}^i \lambda_k \left[ (k+\delta+\lambda_1)(k+\delta+\lambda_2) \cdots (k+\delta+\lambda_i) \right]^{-1}$$

$$= \sum_{i=N+1}^{M-1} T_i (h(i) - h(i+1)) - T_N h(N+1) + T_M h(M),$$

where $h(i) = \exp (-\phi(l_i)) \cdot \left[ \prod_{k=1}^i (1+(k+\delta)/\lambda_k) \right]^{-1}$. By arguments similar to those employed before we may write

$$h(i) = K \cdot g(i)$$

for $i > n_1$, 

where
(i) \[ K = \left[ \prod_{n=1}^{m} (1 + (k + \delta)/\lambda_n) \cdot \exp \left(-\frac{(k + \delta)}{\lambda_n}\right) \right]^{-1}, \]

(ii) \[ g(i) = \exp \left\{ -(\phi(i) + \phi(k + \delta + \epsilon_i)) \right\}, \]

(iii) \[ \lim_{i \to \infty} \epsilon_i = 0, \]

(iv) \[ n_1 \text{ is a sufficiently large integer.} \]

Accordingly, by (4.8), (4.9), and (4.10) we obtain

\[
| \sum_{i=i}^{M} a_i [\lambda_1 \cdot \lambda_i][(k+\delta+\lambda_1)\ldots(k+\delta+\lambda_i)]^{-1} |
\leq |K| \left\{ \sum_{i=0}^{M-1} U_i |g(i) - g(i+1)| + U_N g(N+1) + U_M g(M) \right\}.
\]

On the other hand, for sufficiently large \( i \), we get easily

\[
| g(i) - g(i+1) | = O\left( \int_{i}^{i+1} \frac{d}{dx} \exp \left(-\left(\phi(x) + x(k + \delta)\right)\right) dx \right)
= O\left( \frac{1}{U_i} \int_{i}^{i+1} \exp \left(-\frac{\delta}{2} \cdot x\right) |\phi'(x)| dx \right).
\]

Hence, by (4.9), (4.10) and the hypothesis (d) part (ii), we get for sufficiently large \( N \)

\[
\sum_{i=0}^{M} a_i [\lambda_1 \cdot \lambda_i][(k+\delta+\lambda_1)\ldots(k+\delta+\lambda_i)]^{-1}
= O\left( \int_{i}^{i+N} \exp \left(-\frac{\delta}{2} \cdot x\right) |\phi'(x)| dx \right) + O(\exp(-i_{N+1}(\delta/2 + \epsilon_{N+1})))
+ O(\exp(-i_M(\delta/2 + \epsilon_M))) = o(1),
\]

so that (1.1) is simply convergent at \( s = k + \delta \). Therefore

\[ \sigma_s < k + \delta \]

for any given \( \delta > 0 \), which proves (4.6).

Thus, by (4.3), (4.7), and Theorem 1, we have

\[ k = \sigma_s = \sigma_m, \]

which proves (a) of Theorem III. By the slight modification of the above arguments, we can also prove (b) of Theorem III.

5. Proof of corollaries. By M. Fujiwara's theorem [8], the simple convergence-abscissa \( \sigma_s(G) \) and the absolute convergence-abscissa
\( \sigma_a(G) \) of \( G(s) \) are given respectively by

\[
\sigma_a(G) = \limsup_{n \to \infty} 1/n \cdot \log \left| \sum_{r=1}^{n} a_r \exp \left( \frac{s^2}{n} - \frac{s^2}{r} \right) \right|
\]

(5.1)

\[
\sigma_a(G) = \limsup_{n \to \infty} 1/n \cdot \log \left\{ \sum_{r=1}^{n} \left| a_r \exp \left( \frac{s^2}{n} - \frac{s^2}{r} \right) \right| \right\}
\]

Since \( \phi(x) = x^2 \) evidently satisfies the conditions of Theorem III, taking account of Theorem III and (5.1) we get

\[
\sigma_a = \sigma_a(G), \quad \sigma_a = \sigma_a(G),
\]

which proves Corollary I.

By T. Kojima's theorem [9], we may write

\[
\sigma_a(G) = \limsup_{x \to \infty} 1/x \cdot \log \left| \sum_{\lfloor x \rfloor \leq \nu < x} a_{\nu} \right|
\]

\[
\sigma_a(G) = \limsup_{x \to \infty} 1/x \cdot \log \left\{ \sum_{\lfloor x \rfloor \leq \nu < x} | a_{\nu} | \right\}
\]

so that the first part of Corollary II follows immediately from Corollary I. On the other hand, by a well known theorem [10, p. 49], we have

\[
0 \leq \sigma_a(G) - \sigma_a(G) \leq \limsup_{n \to \infty} 1/n \cdot \log n,
\]

which proves the second part of corollary II.

REFERENCES