ON THE TOTAL VARIATION OF SOLUTIONS OF THE BOUNDED VARIATION MOMENT PROBLEM[1]

MELVIN BLOOM

1. Introduction. In this paper we consider functions $\phi(t)$ which satisfy for a given real sequence $\{\mu_n\}$ the equations

$$(\mu, a) \qquad \mu_n = \int_a^\infty t^n d\phi(t), \qquad n = 0, 1, 2, \cdots,$$

where the integrals converge absolutely. Any such function $\phi(t)$ is called a solution of the moment problem (μ, a) or simply a solution of (μ, a).

Boas [1][2] first pointed out that for an arbitrary sequence $\{\mu_n\}$ there exist infinitely many solutions of $(\mu, 0)$; in fact, he showed that any such sequence can be decomposed in infinitely many ways into the difference of two (Stieltjes) sequences $\{\lambda_n\}$ and $\{\nu_n\}$ where both $(\lambda, 0)$ and $(\nu, 0)$ have nondecreasing solutions. In this theorem the choice of λ_0 and ν_0 is subject only to the conditions

$$\lambda_0 > 0, \qquad \nu_0 > 0, \qquad \lambda_0 - \nu_0 = \mu_0.$$

For arbitrary $\epsilon > 0$ we can choose $\lambda_0 = \mu_0 + \epsilon/2$, $\nu_0 = \epsilon/2$, or $\lambda_0 = \epsilon/2$, $\nu_0 = -\mu_0 + \epsilon/2$, according as $\mu_0 \geqq 0$ or $\mu_0 < 0$, and it follows that there exists a solution $\phi(t)$ of $(\mu, 0)$ having total variation less than $|\mu_0| + \epsilon$.

If $\psi(t)$ is a solution of $(\lambda, 0)$, $\lambda_n = \sum_0^\infty (-1)^i C_{n,i} a^i \mu_n$, a arbitrary, such that the total variation of $\psi(t)$ is less than $|\lambda_0| + \epsilon$, then $\phi(t) = \psi(t-a)$ is a solution of (μ, a) with total variation less than $|\mu_0| + \epsilon$. The proof of the following theorem is now easy.

THEOREM 1.1. *Let $\psi(t)$ be a solution of $(\lambda, -\infty)$ and $\{\mu_n\}$ be a sequence with $\mu_0 = \lambda_0$. Then for arbitrary a and $\epsilon > 0$ there exists a $\phi(t)$ satisfying*

$$\mu_n = \int_{-\infty}^a t^n d\psi(t) + \int_a^\infty t^n d(\psi(t) + \phi(t)), \qquad n = 0, 1, 2, \cdots,$$

$$\int_{-\infty}^\infty |d\phi(t)| < \epsilon.$$

Received by the editors April 14, 1952.

[1] The material in this paper constitutes a part of a thesis submitted to Northwestern University and prepared under the direction of Professor Walter T. Scott.

[2] Numbers in brackets refer to references at the end of this paper.

Pólya [2] has shown that there are infinitely many entire transcendental solutions of $(\mu, -\infty)$ and [3] that there are infinitely many step function solutions with discontinuities restricted to an arbitrarily preassigned set of points with no finite limit points. In §2, using the method of Pólya, we show that there exists an entire transcendental solution of $(\mu, 0)$ with total variation on the whole real axis arbitrarily near $|\mu_0|$, and in §3, using Pólya's method, we do the same for the step functions. In §4 we give a method for constructing such a step function with discontinuities restricted to the points a, $a^2, a^3, \cdots; a \geqq 2$.

2. Entire transcendental solutions.

THEOREM 2.1. *For $\epsilon > 0$ there exists an entire transcendental solution $\phi(t)$ of $(\mu, 0)$ such that $\int_{-\infty}^{\infty} |d\phi(t)| < |\mu_0| + \epsilon$.*

We shall use the following lemma.

LEMMA 2.1. *For any positive integer n, any real $A' \neq 0$, and any $\epsilon' > 0$, there exists an entire transcendental function $g(z)$ satisfying the following three conditions:*

$$\int_0^{\infty} t^j g(t)dt = \begin{cases} 0, & j = 0, 1, \cdots, n-1, \\ A', & j = n, \end{cases} \tag{2.1}$$

$$\int_{-\infty}^{\infty} t^i |g(t)| \, dt < \epsilon', \qquad i = 0, 1, \cdots, n-1,$$

$$|g(z)| < \epsilon', \qquad |z| \leqq n.$$

We remark that this lemma is the same as Pólya's [2] except for the interval of integration in (2.1). He uses e^{-z^2} as the basis for the construction of the function of his theorem. We now prove the lemma.

For fixed real $\lambda \neq 0, 1$ define

$$f(z) = \frac{(-1)^n 2\lambda}{\pi^{1/2} n!(\lambda - 1)} \frac{d^n}{dz^n} (e^{-z^2} - e^{-\lambda^2 z^2}).$$

Then $|f(z)| < M < \infty$, $|z| \leqq 1$,

$$\int_0^{\infty} t^j f(t)dt = \begin{cases} 0, & j = 0, 1, \cdots, n-1, \\ 1, & j = n, \end{cases}$$

$$\int_{-\infty}^{\infty} |t^i| \, |f(t)| \, dt < L < \infty, \qquad j = 0, 1, \cdots, n-1.$$

Choose $\alpha > 0$ so that $|A'| L\alpha < n$, $|A'| M\alpha^{n+1} < \epsilon'$, and $n\alpha < 1$. Then

for $\beta = A'\alpha^{n+1}$, $\beta f(\alpha z)$ satisfies the three conditions of the lemma.

To prove the theorem let $g_0(z) = [2\lambda\mu_0/\pi^{1/2}(\lambda-1)](e^{-z^2} - e^{-\lambda^2 z^2})$ and for indices $n = 1, 2, 3, \cdots$ use the lemma with A' and ϵ' replaced by A_n and $2^{-n}\epsilon$ to choose recursively the functions $g_n(z)$ satisfying

$$\int_0^\infty t^j g_n(t)dt = \begin{cases} 0, & j = 0, 1, \cdots, n-1, \\ A_n, & j = n, \end{cases}$$

where

$$A_n = \mu_n - \int_0^\infty t^n(g_0(t) + \cdots + g_{n-1}(t))dt.$$

It is readily verified that any $\phi(t)$ for which $\phi'(t) = \sum_0^\infty g_i(t)$ satisfies the conditions of the theorem.

3. Step function solutions. Here we neglect μ_0. This is no loss, since a solution for the sequence $\mu_0, \mu_1, \mu_2, \cdots$ becomes a solution for the sequence $\mu_0', \mu_1, \mu_2, \cdots$ by the addition at the origin of a discontinuity with saltus $(\mu_0' - \mu_0)$.

THEOREM 3.1. *If the sequence $0 < b_1 < b_2 < \cdots$ has limit point infinity, then for $\epsilon > 0$ and any sequence $\{\mu_n\}$ there exists a solution of the infinite system*

$$\sum_{i=1}^\infty b_i^n u_i = \mu_n, \qquad n = 1, 2, 3, \cdots,$$

satisfying the following two conditions:

$$\sum_{i=1}^\infty b_i^n |u_i| < \infty, \qquad n = 1, 2, 3, \cdots,$$

$$\sum_{i=1}^\infty |u_i| < \epsilon.$$

This theorem is a corollary of

THEOREM 3.2. *Let the matrix $A = (a_{jk})$ with complex elements satisfy the following three conditions*:

$$(3.1) \qquad \begin{cases} \textit{Every segment of the form} \\ \quad a_{1,q+1}, a_{1,q+2}, \cdots, \\ \quad \cdots\cdots\cdots, \\ \quad a_{n,q+1}, a_{n,q+2}, \cdots \\ \textit{is of rank } n, \end{cases} \qquad n = 2, 3, 4, \cdots,$$

$$\lim_{k\to\infty} \frac{a_{j-1,k}}{a_{j,k}} = 0, \qquad j = 2, 3, 4, \cdots, \tag{3.2}$$

$$\limsup_{k\to\infty} |a_{1k}| = \infty. \tag{3.3}$$

Then for arbitrary $\epsilon > 0$ *and arbitrary* $\mu = (\mu_1, \mu_2, \mu_3, \cdots)$ *there exists a* $u = (u_1, u_2, u_3, \cdots)$ *satisfying the following three conditions*:

$$Au = \mu, \tag{3.4}$$

$$\sum_{k=1}^{\infty} |a_{jk}u_k| < \infty, \qquad j = 1, 2, 3, \cdots, \tag{3.5}$$

$$\sum_{k=1}^{\infty} |u_k| < \epsilon. \tag{3.6}$$

Condition (3.3) *is necessary for condition* (3.6).

The following lemma will be used in the proof of the theorem.

LEMMA 3.1. *Let the matrix* (a_{jk}) *satisfy conditions* (3.1), (3.2), *and* (3.3). *Then for arbitrary* $\epsilon' > 0$, *arbitrary positive integers* n *and* q, *and arbitrary number* μ', *there exists an integer* $q' > q + n$ *and numbers* $u_{q+1}, u_{q+2}, \cdots, u_{q'}$ *satisfying the following three conditions*:

$$a_{j,q+1}u_{q+1} + \cdots + a_{jq'}u_{q'} = \begin{cases} 0, & j = 1, 2, \cdots, n-1, \\ \mu', & j = n, \end{cases} \tag{3.7}$$

$$|a_{j,q+1}u_{q+1}| + \cdots + |a_{jq'}u_{q'}| < \epsilon', \qquad j = 1, 2, \cdots, n-1, \tag{3.8}$$

$$|u_{q+1}| + \cdots + |u_{q'}| < \epsilon'. \tag{3.9}$$

We remark that Pólya [3] proves Theorem 3.2 with conditions (3.3) and (3.6) deleted. His proof depends upon a lemma which is Lemma 3.1 with conditions (3.3) and (3.9) deleted.

We now prove the lemma.

Condition (3.1) implies the existence of a set of indices $k_1, k_2, \cdots, k_n$, $q < k_1 < k_2 < \cdots < k_n$ for which the determinant of the system

$$a_{jk_1}u_{k_1} + \cdots + a_{jk_n}u_{k_n} = x_j, \qquad j = 1, 2, \cdots, n, \tag{3.10}$$

does not vanish and hence there exists a $\delta > 0$ such that

$$|a_{jk_1}u_{k_1}| + \cdots + |a_{jk_n}u_{k_n}| < \min\left(\frac{\epsilon'}{2}, \frac{M\epsilon'}{2n}\right) \tag{3.11}$$

for $|x_j| < \delta$, $j = 1, 2, \cdots, n$, where M is the smallest of the absolute values of the nonzero elements in the coefficient matrix of the system (3.10). The number δ is now fixed.

Condition (3.3) implies the existence of an infinite sequence of indices $l_1<l_2<l_3<\cdots$ for which $|a_{1l_i}|\geqq 1$. Condition (3.2) gives the existence of an index K such that for $k>K$,

$$\frac{|\mu' a_{jk}|}{|a_{nk}|}<\min\left(\delta, \frac{\epsilon'}{2}, \frac{M\epsilon'}{2n}\right), \qquad j=1, 2, \cdots, n-1.$$

Let $q'=l_i$ for some $l_i>\text{Max}\ (K, k_n)$ so that

$$\frac{|\mu'|}{|a_{nq'}|}<\frac{|\mu' a_{jq'}|}{|a_{nq'}|}, \qquad j=1, 2, \cdots, n-1.$$

With δ and q' now fixed put $x_n=0$ and

$$x_j=-\frac{\mu' a_{jq'}}{a_{nq'}}, \qquad j=1, 2, \cdots, n-1,$$

$$u_{q'}=\frac{\mu'}{a_{nq'}},$$

$$u_k=0, \qquad q<k<q',\ k\neq k_1, k_2, \cdots, k_n.$$

It is readily verified that the numbers $u_{q+1}, u_{q+2}, \cdots, u_{q'}$ so determined satisfy conditions (3.7), (3.8), and (3.9).

We turn to the proof of the theorem and consider first the necessity of condition (3.3) for condition (3.6). If $\limsup_{k\to\infty}|a_{1k}|<\infty$ the sequence $\{|a_{1k}|\}$ has an upper bound B. The sequence $\{\mu_i\}$ is arbitrary and, for $\mu_1\neq 0$, there exists an n for which $|\sum_1^n a_{1k}u_k-\mu_1|<|\mu_1|/2$. Then $\sum_{k=1}^n |u_k|>|\mu_1|/2B$ and condition (3.6) cannot be satisfied.

To complete the proof of the theorem we note that condition (3.3) implies the existence of an index q_1 for which $|a_{1q_1}|>2|\mu_1|/\epsilon$. Let

$$u_k=\begin{cases}0, & k=1, 2, \cdots, q_1-1,\\ \mu_1/a_{1q_1}, & k=q_1.\end{cases}$$

Then

$$\sum_{k=1}^{q_1} a_{jk}u_k=\mu_1,$$

and

$$\sum_{k=1}^{q_1}|u_k|<\epsilon/2.$$

This completes the first step.

At the nth step, $n>1$, we let μ', ϵ', n, q, and q' of the lemma be

$\mu_n - \sum_{j=1}^{q_{n-1}} a_{nj}u_j$, $2^{-n}\epsilon$, n, q_{n-1}, and q_n respectively and we have

$$\sum_{k=q_{n-1}+1}^{q_n} a_{jk}u_k = \begin{cases} 0, & j = 1, 2, \cdots, n-1, \\ \mu_n, & j = n, \end{cases}$$

$$\sum_{k=q_{n-1}+1}^{q_n} | a_{jk}u_k | < 2^{-n}\epsilon, \quad j = 1, 2, \cdots, n-1,$$

$$\sum_{k=q_{n-1}+1}^{q_n} | u_k | < 2^{-n}\epsilon.$$

Combining the results of the first n steps we have

$$\sum_{k=1}^{q_n} a_{jk}u_k = \mu_j, \qquad j = 1, 2, \cdots, n,$$

$$\sum_{k=1}^{q_n} | a_{jk}u_k | < \sum_{k=1}^{q_j} | a_{jk}u_k | + \epsilon, \qquad j < n,$$

$$\sum_{k=1}^{q_n} | u_k | < \epsilon.$$

These conditions hold for all n, and this completes the proof of the theorem.

We now prove a theorem to be used in §4.

THEOREM 3.3. *If for the points $b_1, b_2, b_3, \cdots$ we have $b_{i+1} - b_i \geqq b > 0$, $i = 1, 2, 3, \cdots$, then every solution $u_1, u_2, u_3 \cdots$ of the equations*

$$\sum b_i^n u_i = \mu_n, \qquad n = 1, 2, 3, \cdots,$$

has absolutely convergent sums in the left member.

Suppose for some integer m

$$\sum_{i=1}^{\infty} b_i^m | u_i | = \infty.$$

Then for an infinite sequence of indices $l_2 < l_3 < l_4 < \cdots$,

$$b_{l_i}^m | u_{l_i} | > \frac{1}{i(\log i)^{n/2}}, \qquad i = 3, 4, \cdots,$$

$$b_{l_i}^{m+2} | u_{l_i} | > \frac{l_i^2 b^2}{i(\log i)^{3/2}}, \qquad i = 3, 4, \cdots,$$

$$b_{l_i}^{m+2} | u_{l_i} | > b^2, \qquad i = 3, 4, \cdots.$$

Hence the sums diverge for $n \geqq m+2$, contradicting the hypothesis.

For the example $u_i = (-1)^i/i(\log i)^{3/2}$, $b_i = \log i$, the sums converge conditionally, although $\sum_{i=1}^{\infty} |u_i|$ converges.

4. **An example.** For $(\mu, 0)$, $\mu_0 = 0$, and for arbitrary $\epsilon > 0$, $a \geqq 2$, we here construct a step function solution $\phi(t)$ whose discontinuities are all at the points $a, a^2, a^3, \cdots$, and whose total variation is less than ϵ.

Consider, for any fixed integer r, the infinite system

$$
\begin{aligned}
\delta_{r0} &= u_1 \\
0 &= \qquad u_2 + u_3 \\
\delta_{r1} &= u_1 + au_2 + a^2u_3 \\
0 &= \qquad\qquad\qquad u_4 + u_5 + u_6 \\
0 &= \qquad\qquad\qquad u_4 + au_5 + a^2u_6 \\
\delta_{r2} &= u_1 + a^2u_2 + a^4u_3 + a^6u_4 + a^8u_5 + a^{10}u_6 \\
&\qquad\qquad\qquad\qquad \vdots
\end{aligned}
\tag{4.1}
$$

where, if $\sigma_k = k(k-1)/2$, the (σ_{k+1})th equation is

$$\delta_{r,k-1} = \sum_{i=1}^{\sigma_{k+1}} a^{(i-1)(k-1)}u_i \equiv P_{r,k-1}, \qquad k = 1, 2, 3, \cdots, \tag{4.2}$$

and the $(\sigma_{k+1}+l)$th equation is

$$0 = \sum_{i=1}^{k+1} a^{(i-1)(l-1)}u_{\sigma_{k+1}+i} \equiv Q_{rkl}, \qquad l = 1, 2, \cdots, k. \tag{4.3}$$

THEOREM 4.1. *For any integer r and any $a \neq 1$ the system* (4.1) *has a unique solution $u_1, u_2, u_3, \cdots$. For $|a| > 1$ this solution satisfies the system*

$$\sum_{n=1}^{\infty} a^{(n-1)h}u_n = \delta_{rh}, \qquad h = 0, 1, 2, \cdots, \tag{4.4}$$

the sums converging absolutely.

Let the determinant $A_{nk} \equiv |\alpha_{ij}|$, $k \geqq n-1$, where $\alpha_{ij} = a^{(i-1)(j-1)}$, $i = 1, 2, \cdots, n-1$, and $\alpha_{in} = a^{k(n-1)}$. Let $A_{nk}^{(l)}$ be the cofactor of $a^{k(l-1)}$ in A_{nk}. It is readily verified that

$$\frac{A_{n,n-1}^{(l)}}{A_{n,n-1}} = \frac{(-1)^{n+l}}{a^{\sigma_{l-1}+(l-1)(n-l)}(a-1)\cdots(a^{l-1}-1)(a-1)\cdots(a^{n-l}-1)},$$

$$\frac{A_{nk}}{A_{n,n-1}} = \frac{(a^{k-n+2}-1)\cdots(a^k-1)}{(a-1)\cdots(a^{n-1}-1)}.$$

For any r define $S_j=\delta_{r,j-1}$, $j\leqq r+1$, and $S_j=-\sum_1^{\sigma_j} a^{(i-1)(j-1)}u_i$ for $j>r+1$. Applying the above determinant formulas to system (4.1) we have for $j=1, 2, 3, \cdots,$ and $i=1, 2, \cdots, j$,

$$(4.5)\quad u_{\sigma_j+i}=\frac{(-1)^{i+j}S_j}{a^{\sigma_j(j-1)+(j-i)(i-1)+\sigma_{i-1}}(a-1)\cdots(a^{i-1}-1)(a-1)\cdots(a^{j-i}-1)},$$

$$(4.6)\quad S_j=-\sum_{l=r+1}^{j-1} a^{\sigma_l(j-l)}\frac{(a^{j-l+1}-1)\cdots(a^{j-1}-1)}{(a-1)\cdots(a^{l-1}-1)}S_l.$$

These equations give the solution of the system (4.1) for $a\neq 1$.

Using the fact that the fraction in the coefficient of S_l in (4.6) reduces to a polynomial of degree $(j-l)(l-1)$ with the sum of its coefficients $C_{j-1,l-1}$, we can establish the inequality

$$(4.7)\qquad |S_j| < |a|^{\sigma_{j-1}(j+3)/3}[(j-1)!]^{3/2}, \qquad j\geqq 4,\ |a|>1.$$

This, with (4.5), shows that the sums in (4.4) converge absolutely for $|a|>1$. We have, for any r, using (4.2) and (4.3),

$$\sum_{n=1}^{\infty} a^{(n-1)h}u_n = P_{rh} + \sum_{i=2}^{\infty} a^{\sigma_{h+i}h}Q_{r,h+i-1,h+1}$$
$$= \delta_{rh},$$

since a subsequence of the partial sums converges to δ_{rh}. This completes the proof of the theorem.

In constructing the function $\phi(t)$ we observe, using (4.5) and (4.7), that for $a\geqq 2$, $|u_n|<2^{-n}$, $n\geqq 7$, so that $\sum|u_n| = \sum_1^6|u_n| + \sum_7^\infty|u_n| <9/8$ for $r\geqq 1$. Put

$$\psi_r(t) = \begin{cases} 0, & 0\leqq t<a, \\ \sum_{i=1}^{k} u_i, & a^{k-1}\leqq t<a^k,\ k=1, 2, 3, \cdots, \end{cases}$$

and then $\int_0^\infty t^n d\psi_r(t)=\delta_{nr}$, $n=1, 2, 3, \cdots,$ and $\int_0^\infty |d\psi_r(t)|<9/8$.

Now for arbitrary $\mu_1, \mu_2, \mu_3, \cdots, a\geqq 2, \epsilon>0$, and for each index $r=1, 2, 3, \cdots,$ let l_r be the smallest positive integer for which $a^{rl_r}>(9/8)\cdot 2^r|\mu_r|\epsilon^{-1}$, and for $\phi_r(t)=\mu_r a^{-rl_r}\psi_r(a^{-l_r}t)$ we have

$$\int_0^\infty t^n d\phi_r(t) = \delta_{rn}\mu_r, \qquad n, r = 1, 2, 3, \cdots,$$

and

$$\int_0^\infty |\, d\phi_r(t) \,| < 2^{-r}\epsilon.$$

Then $\phi(t) = \sum_1^\infty \phi_r(t)$ satisfies

$$\int_0^\infty t^n d\phi(t) = \mu_n, \qquad n = 1, 2, 3, \cdots,$$

and

$$\int_0^\infty |\, d\phi(t) \,| < \epsilon.$$

Moreover the points of change of $\phi(t)$ are all included among the points a, a^2, a^3, $\cdots$, and by Theorem 3.3 the integrals converge absolutely.

References

1. R. P. Boas, *The Stieltjes moment problem for functions of bounded variation*, Bull. Amer. Math. Soc. vol. 45 (1939) pp. 399–404.

2. G. Pólya, *Sur l'indétermination d'un théorème voisin du problème des moments*, C. R. Acad. Sci. Paris vol. 207 (1938) pp. 708–711.

3. ———, *Eine einfache, mit functionen theoretischen Aufgaben verknüpfte, hinreichende Bedingung für die Auflösbarkeit eines Systems unendlich vieler linearer Gleichungen*, Comment. Math. Helv. vol. 11 (1938) pp. 234–252.

Miami University and
Northwestern University