LINE INTEGRAL APPROXIMATION OF DOUBLE INTEGRALS

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1. Introduction. It has been shown by Wilkins [2] and Grosswald [1] that the double integral of a continuous function $F(x, y)$ over a circular region centered at the origin is the limit of a related integral along the spiral $\theta = \alpha r$ as $\alpha \to \infty$. This result is generalized in several directions in the present paper.

The authors were led to this study by way of certain special cases arising in connection with the design of an analog computer at Sandia Corporation. This device among other things approximates the probability mass

$$\frac{1}{2\pi}\iint \exp\,(-r^2/2) r\,dr\,d\theta, \tag{1.1}$$

for a wide variety of regions R, by scanning R along arcs of a related spiral.

2. A general limit theorem. Let R be a bounded, measurable subset of the plane. Let $f(r, \theta)$, $g(r, \theta)$, and $h(r)$ be functions with the following properties:

(2.1) *$f(r, \theta)$ is continuous on the closure $\overline{R}$ of R, and periodic in θ of period 2π.*

(2.2) *$h(r)$ is positive, almost everywhere on $[0, a]$, and is summable on $[0, a]$, where a is the radius of a circle, centered at the origin, containing $\overline{R}$ in its interior.*

It is convenient to the exposition to set $f(r, \theta) \equiv 0$, $(r, \theta) \notin \overline{R}$, and $h(r) \equiv 1$, $r > a$.

(2.3) *$g(r, \theta) \equiv h(r) f(r, \theta)$, and hence is of period 2π in θ.* It follows that $|g|$ is dominated on R by a function summable over R and hence that g is summable over R.

We consider the integral

$$I = \frac{1}{2\pi}\iint g(r, \theta)\,dr\,d\theta, \tag{2.4}$$

subject to conditions (2.1) to (2.3).

For $\alpha > 0$, let Γ_α denote the spiral

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$$\theta = \alpha \int_0^r h(u)du, \tag{2.5}$$

and define a sequence $\{r_i\}$ by the equation

$$2i\pi = \alpha \int_0^{r_i} h(u)du, \qquad i = 0, 1, 2, \cdots. \tag{2.6}$$

The set of points (r, θ) satisfying inequalities $r_i \leqq r \leqq r_{i+1}$ will be denoted by S_i. Let C_i denote any of the class of arcs in S_i on each of which r is nondecreasing in θ $(0 \leqq \theta \leqq 2\pi)$. Let

$$Q_i^+ \equiv \sup_{C_i} \int_{C_i \cap R} f(r, \theta)d\theta, \quad \text{and} \quad Q_i^- \equiv \inf_{C_i} \int_{C_i \cap R} f(r, \theta)d\theta. \tag{2.7}$$

For each $r > 0$, let E_r denote the set of θ-values corresponding to all points (r, θ) in R. As a consequence of (2.3) and Fubini's theorem, we have

$$I = \frac{1}{2\pi} \int_0^\infty h(r)dr \int_{E_r} f(r, \theta)d\theta. \tag{2.8}$$

Our approximating line integral is

$$I^* \equiv \frac{1}{\alpha} \int_{\Gamma_\alpha \cap R} f(r, \theta)d\theta = \int_{\Gamma_\alpha \cap R} g(r, \theta)dr. \tag{2.9}$$

Let

$$I_i \equiv \frac{1}{2\pi} \int_{r_i}^{r_{i+1}} h(r)dr \int_{E_r} f(r, \theta)d\theta, \tag{2.10}$$

and

$$I_i^* \equiv \frac{1}{\alpha} \int_{\Gamma_\alpha \cap R \cap S_i} f(r, \theta)d\theta = \int_{\Gamma_\alpha \cap R \cap S_i} g(r, \theta)dr. \tag{2.11}$$

Clearly

$$I = \sum_0^\infty I_i = \sum_0^{m-1} I_i \quad \text{and} \quad I^* = \sum_0^\infty I_i^* = \sum_0^{m-1} I_i^* \tag{2.12}$$

where $m = m(\alpha)$ is the largest integer such that $r_m \leqq a$. We consider only values of α such that R is in the circle of radius r_m centered at the origin. It follows from definitions (2.7) and (2.11) that

$$Q_i^- \leqq \alpha I_i^* \leqq Q_i^+.$$

We see from (2.6) and (2.10) that I_i satisfies these same inequalities and hence that

$$(2.13) \qquad \left| I_i - I_i^* \right| \leqq \frac{1}{\alpha} (Q_i^+ - Q_i^-).$$

It then follows from (2.12) that

$$(2.14) \qquad \left| I - I^* \right| \leqq \frac{1}{\alpha} \sum (Q_i^+ - Q_i^-).$$

Given $\epsilon > 0$ and i, a positive integer or zero, definitions (2.7) imply the existence of functions $\lambda_i(\theta)$ and $\mu_i(\theta)$, each bounded by r_i and r_{i+1} and monotone nondecreasing on $[0, 2\pi]$, such that

$$(2.15) \qquad \left| Q_i^+ - \int_0^{2\pi} f[\lambda_i(\theta), \theta]\phi_i(\theta) d\theta \right| < \epsilon,$$

and

$$(2.16) \qquad \left| Q_i^- - \int_0^{2\pi} f[\mu_i(\theta), \theta]\psi_i(\theta) d\theta \right| < \epsilon,$$

in which

$$\begin{aligned} \phi_i(\theta) &= 1 \text{ if } [\lambda_i(\theta), \theta] \in R \\ &= 0 \text{ otherwise,} \end{aligned}$$

and similarly for $\psi_i(\theta)$. It follows that

$$(2.17) \qquad Q_i^+ - Q_i^- < 2\epsilon + \int_0^{2\pi} \Delta(\theta) d\theta,$$

where $\Delta \equiv \left| f[\lambda_i(\theta), \theta]\phi_i(\theta) - f[\mu_i(\theta), \theta]\psi_i(\theta) \right|$.

For fixed i, let s_i denote the set of θ-values for which either $(r, \theta) \in R$ for each r on $[r_i, r_{i+1}]$, or $(r, \theta) \notin R$ for each r on $[r_i, r_{i+1}]$. Thus, for $\theta \in s_i$, either both $(\lambda_i(\theta), \theta)$ and $(\mu_i(\theta), \theta)$ are in R, or neither is. Let s_i' be the complementary set, $[0, 2\pi] - s_i$. It follows from (2.1) and from (2.2) and (2.6), which imply that $(r_{i+1} - r_i) \to 0$ uniformly with respect to i as $\alpha \to \infty$, that there exists a positive number A such that

$$(2.18) \qquad \Delta < \epsilon, \qquad \text{for } \theta \in s_i, \text{ if } \alpha > A, \text{ independently of } i.$$

Let M be a bound on $|f(r, \theta)|$ for $(r, \theta) \in R$. Then

$$(2.19) \qquad \Delta \leqq 2M \qquad \text{for arbitrary } \theta, i.$$

It follows from (2.6), (2.13), (2.17), (2.18), and (2.19) that, for $\alpha > A$,

$$|I_i - I_i^*| < \left[\frac{\epsilon}{\pi} + \frac{1}{2\pi}\int_{s_i}\Delta d\theta + \frac{1}{2\pi}\int_{s_i'}\Delta d\theta\right]\cdot\int_{r_i}^{r_{i+1}} h(r)dr,$$

and hence that

$$(2.20)\qquad |I_i - I_i^*| < \epsilon\left(\frac{1}{\pi} + 1\right)\int_{r_i}^{r_{i+1}} h(r)dr + \frac{2M}{\alpha} m(s_i'),$$

where $m(s_i')$ denotes the measure of the set s_i'. The bounded set R is inside the circle of radius a with center at the origin. It follows from (2.12) and (2.20) that

$$(2.21)\qquad |I - I^*| \leqq 2\epsilon\int_0^a h(r)dr + \frac{2M}{\alpha}\sum m(s_i').$$

Corresponding to each partition P: $0 = r_0 < r_1 < r_2 < \cdots < r_n = a$ of $[0, a]$ is a sum $\sum m(s_i')$. We now impose upon R the condition that

$$(2.22)\qquad \sup_P \sum m(s_i') < \infty.$$

The following theorem is then a consequence of (2.21).

THEOREM 2.1. *If R is a bounded measurable set with property* (2.22), *then* $I = \lim I^*$ *as* $\alpha \to \infty$.

It is clear from the definition of the sets s_i that if no point of the boundary of R is in S_i and on a given ray from the origin, then the θ-value corresponding to that ray belongs to s_i. Thus s_i' is contained in the set of θ-values subtended at the origin by the part of the boundary of R in S_i. It is then clear that if R is a region on the boundary of which θ is of bounded variation, then R satisfies condition (2.22). In particular, if the boundary of the region R is rectifiable, and if the origin is not on the boundary (from which it follows that θ is of bounded variation), then R has property (2.22).

If the origin O is on the rectifiable boundary, C, of R, and if an arc of C containing O in its interior is a straight line, then also it is clear that θ is of bounded variation. Suppose then that C is rectifiable, that O is on C, and that no arc of C containing O is rectilinear. To each positive number ϵ corresponds a positive number δ such that all points at distance less than δ along C from O lie in a circle of center O and radius ϵ. Choose two such points, P_1 and P_2, on opposite sides of O and not on the same straight line with O; form the boundary of a region R' bounded by the rectifiable curve C' which is C, with the arc P_1P_2 replaced by the straight line P_1P_2. Since O is not on the boundary of R', Theorem 2.1 applies to R'. From (2.1)–(2.3), (2.8), and (2.9), it is clear that the integrals I and I^* corresponding to regions contained in a circle of radius ϵ are arbitrarily small for sufficiently small ϵ. It follows that $\lim_{\alpha\to\infty} I^* = I$ for R, since $R-R'$ lies in the circle with center O and radius ϵ. We have then the following corollary of Theorem 2.1:

If R is bounded by a finite set of rectifiable simple closed curves, then $I = \lim I^*$ as $\alpha \to \infty$, irrespective of the choice of origin.

3. Integrands in which variables are separable. We now consider the special case in which f is independent of r. We retain hypotheses (2.1) to (2.3) and assume further that $f(\theta) \geqq 0$.

A region bounded by two rays from the origin and an arc with polar equation $r = F(\theta)$, $F(\theta)$ continuous and single-valued, will be called a *wedge*. If $F(\theta)$ and $G(\theta)$ are both continuous and single-valued with $0 \leqq F(\theta) \leqq G(\theta)$, the region bounded by two rays from the origin and by arcs with respective equations $r = F(\theta)$ and $r = G(\theta)$ will be called a *truncated wedge*. If $F(\theta) \equiv 0$, the truncated wedge becomes a wedge. The two bounding segments of rays are *sides* of a wedge or truncated wedge, and the angle between 0 and 2π subtended by the boundary is the *angle* of the wedge or truncated wedge. The point of intersection of the sides is the *vertex*.

LEMMA 3.1. *If R is a wedge of angle ϕ with sides $\theta = \theta_1$, and $\theta = \theta_1 + \phi$, and if k is any positive integer or zero, then*

$$\sum_0^k (Q_i^+ - Q_i^-) \leqq \int_{\theta_1}^{\theta_1+\phi} f(\theta)d\theta.$$

PROOF. It follows from definition (2.7), the definition of the sets E_r, and the single-valuedness of $F(\theta)$, that

$$Q_i^- = \int_{E_{i+1}} f(\theta)d\theta, \quad \text{and} \quad Q_i^+ = \int_{E_i} f(\theta)d\theta. \tag{3.1}$$

Here and elsewhere in this section E_i means E_r, $r = r_i$.

The set E_r for a wedge is monotone nonincreasing in r since $F(\theta)$ is single-valued. Thus E_r has a limit as $r \to 0$. This limit defines E_0 appearing in the second integral (3.1) when $i = 0$. We find by addition that

$$\sum_0^k (Q_i^+ - Q_i^-) = Q_0^+ - Q_{k+1}^- = \int_{E_0} f(\theta)d\theta - \int_{E_{k+1}} f(\theta)d\theta.$$

The first term on the right does not exceed the right member of the relation we wish to prove. Hence the lemma is true as stated.

LEMMA 3.2. *If R is a truncated wedge of angle ϕ, with sides $\theta=\theta_1$, and $\theta=\theta_1+\phi$, if there exists a number r^* such that $F(\theta)\leqq r^*\leqq G(\theta)$, and if k is any positive integer or zero, then*

$$\sum_0^k (Q_i^+ - Q_i^-) \leqq 2\int_{\theta_1}^{\theta_1+\phi} f(\theta)d\theta.$$

PROOF. Let j denote the largest non-negative integer such that $r_j\leqq r^*$. The set E_r is monotone nondecreasing in r for $0\leqq r\leqq r^*$ but monotone nonincreasing for $r>r^*$. It follows that

$$\begin{aligned} Q_i^- &= \int_{E_i} f(\theta)d\theta, && i<j,\\ &= \min\left\{\int_{E_i} f(\theta)d\theta,\ \int_{E_{i+1}} f(\theta)d\theta\right\}, && i=j,\\ &= \int_{E_{i+1}} f(\theta)d\theta, && i>j, \end{aligned}$$

while

$$\begin{aligned} Q_i^+ &= \int_{E_{i+1}} f(\theta)d\theta, && i<j,\\ &= \int_{E_{r^*}} f(\theta)d\theta, && i=j,\\ &= \int_{E_i} f(\theta)d\theta, && i>j. \end{aligned}$$

If Q_j^- has the first alternative value, we find that

$$\begin{aligned} \sum_0^k (Q_i^+ - Q_i^-) &= \int_{E_{r^*}} + \int_{E_{j+1}} - \int_{E_0} - \int_{E_{k+1}} f(\theta)d\theta, && k>j,\\ &= \int_{E_{k+1}} - \int_{E_0} f(\theta)d\theta, && k\leqq j. \end{aligned} \tag{3.2}$$

None of the integrals $\int_{E_{r^*}}, \int_{E_{j+1}}, \int_{E_{k+1}} f(\theta)d\theta$ exceeds

$$\int_{\theta_1}^{\theta_1+\phi} f(\theta)d\theta. \tag{3.3}$$

The lemma follows in this case from (3.2), and the assumption that

$f \geqq 0$. If Q_j^- has the second value, the only change is to replace subscript $j+1$ by j in the second integral of (3.2).

LEMMA 3.3. *The interior of every truncated wedge W is the union of a countable set of nonoverlapping truncated wedges corresponding to each of which is a number r^* with the property stated in the preceding lemma. Moreover, the sum of the angles of these truncated wedges does not exceed the angle of W.*

PROOF. If $F(\theta) \equiv G(\theta)$, or if the angle of W is zero, there are no interior points, and the lemma is vacuously true. Suppose next that there is an interior point. Through each interior point (r, θ) of W pass circular arcs of radius r, all points of which are interior to W. Rays from the origin through the ends of such an arc bound, with $r = F(\theta)$ and $r = G(\theta)$ (the bounding arcs of W), a truncated wedge w contained in W. Let I denote the open θ-interval subtended by the interior of w.

Consider the class $\{I\}$ of all such I. By the Lindelöf Covering Theorem, there exists a countable subset of $\{I\}$, which we denote by

$$I_1, I_2, I_3, \cdots, \tag{3.4}$$

which covers the open set of θ-values corresponding to interior points of W. By subtracting the union of preceding intervals from each interval I_n, we obtain a countable set of nonoverlapping intervals. The union of the corresponding truncated wedges is the interior of W. It is clear that the sum of their angles is not greater than the angle of W (it may be less; e.g., if $F \equiv G$ on an interval).

One can show by examples that "countable" cannot be replaced by "finite" in this lemma.

A decomposition of a region R into wedges, or into truncated wedges, will be called *simple* if no two have corresponding θ-intervals with common interior points. It follows that the sum of the angles of the wedges (truncated wedges) of such a decomposition is not greater than 2π. From Lemmas 3.1 to 3.3, relation (2.14), and the observation that both I and I^* are (finitely) additive with respect to disjoint regions R, we have the following theorems:

THEOREM 3.1. *If R admits a simple decomposition into a countable set of wedges with vertices at the origin, then*

$$|I - I^*| \leqq \frac{1}{\alpha} \int_0^{2\pi} f(\theta) d\theta.$$

THEOREM 3.2. *If R admits a simple decomposition into a countable set of truncated wedges with vertices at the origin, then*

$$|I - I^*| \leqq \frac{2}{\alpha}\int_0^{2\pi} f(\theta)d\theta.$$

We have remarked that a wedge is a truncated wedge. Hence Theorem 3.2 applies to mixed decompositions in which some of the truncated wedges reduce to wedges.

We have been considering functions f which are not only independent of r, but are also non-negative. Dropping this restriction, we have the theorem:

THEOREM 3.3. *If R admits a simple decomposition into a countable set of wedges with vertices at the origin, on each of which $f(\theta)$ is of constant sign, then*

$$|I - I^*| \leqq \frac{1}{\alpha}\int_0^{2\pi} |f(\theta)|\, d\theta.$$

The corresponding generalization of Theorem 3.2 is also immediate.

Theorem 3.3 is applicable, for example, to a double integral of a power function $x^m y^n$, when it is transformed into polar coordinates.

It follows from any of the preceding theorems that $I = \lim I^*$ as $\alpha \to \infty$, as has been shown under more general hypotheses in Theorem 2.1.

THEOREM 3.4. *Let R be a circular disc of radius $b < a$. Let the parameter α of (2.6) be such that the point (b, θ_1) is on the spiral Γ_α. A necessary and sufficient condition in order that $I = I^*$, without passage to the limit, is that*

$$\frac{1}{\theta_1}\int_0^{\theta_1} f(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta.$$

In particular, $I = I^*$ if the spiral makes an integral number of turns in R.

PROOF. Let k be the largest integer such that $r_k < b$. Then from (2.10) and (2.11), we have

$$I_i = I_i^* = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \int_{r_i}^{r_{i+1}} h(r)dr, \quad \text{for } 0 \leqq i < k,$$

since, by (2.6),

$$\frac{1}{2\pi}\int_{r_i}^{r_{i+1}} h(r)dr = \frac{1}{\alpha}\cdot$$

Also

$$I_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta \int_{r_k}^{b} h(r)dr,$$

while

$$I_k^* = \frac{1}{2\pi}\int_0^{\theta_1} f(\theta)d\theta \int_{r_k}^{r_{k+1}} h(r)dr.$$

By (2.5),

$$\int_{r_k}^{b} h(r)dr = \frac{\theta_1}{\alpha},$$

while

$$\frac{1}{2\pi}\int_{r_k}^{r_{k+1}} h(r)dr = \frac{1}{\alpha}\cdot$$

Hence

$$I - I^* = I_k - I_k^* = \frac{\theta_1}{2\pi\alpha}\int_0^{2\pi} f(\theta)d\theta - \frac{1}{\alpha}\int_0^{\theta_1} f(\theta)d\theta.$$

The conclusion of the theorem is immediate.

Bibliography

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