NOTE ON DIRICHLET SERIES. IV. ON THE SINGULARITIES OF DIRICHLET SERIES

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Let us put

\[ F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s) \]

\((s = \sigma + it, 0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty).\)

When we vary coefficients \(\{a_n\}\), this change has some influence upon singularities. Concerning this problem, O. Szász [1, p. 107] has proved the next theorem, which is a generalization of Hurwitz-Pólya's theorem [2, p. 36]:

O. Szász's Theorem. Let (1) have the finite simple convergence-abscissa \(\sigma_*\). If \(\lim_{n \to \infty} \log n/\lambda_n = 0\), then there exists a sequence \(\{\epsilon_n\}\) \((\epsilon_n = \pm 1)\) such that \(\sum_{n=1}^{\infty} a_n \epsilon_n \exp(-\lambda_n s)\) has \(\sigma = \sigma_*\) as the natural boundary.

In this note, we shall prove the following theorem of the same type:

Theorem. Let (1) have the finite simple convergence-abscissa \(\sigma_*\). If \(\lim_{n \to \infty} \log n/\lambda_n = 0\), then there exists a Dirichlet series \(\sum_{n=1}^{\infty} b_n \exp(-\lambda_n s)\) having \(\sigma = \sigma_*\) as the natural boundary such that

(a) \(|b_n| = |a_n| \quad (n = 1, 2, \cdots), \text{and} \lim_{n \to \infty} |\arg(a_n) - \arg(b_n)| = 0\)

or

(b) \(\arg(b_n) = \arg(a_n) \quad (n = 1, 2, \cdots), \text{and} \lim_{n \to \infty} |b_n/a_n| = 1.\)

Proof. On account of \(\lim_{n \to \infty} \log n/\lambda_n = 0\), and G. Valiron's theorem [3, p. 4], we get

\[ \sigma_* = \lim \sup_{n \to \infty} \frac{1}{\lambda_n} \log |a_n|. \]

Therefore we can select a subsequence \(\{\lambda_n\}\) such that

(i) \(\sigma_* = \lim \frac{1}{\lambda_n} \log |a_n|, \)

(ii) \(\lim \inf_{i \to \infty} (\lambda_{n+i} - \lambda_n) > 0, \quad \lim_{i \to \infty} i/\lambda_{n+i} = 0. \)

Again by G. Valiron's theorem and (3) (i), \(G_i(s; \theta, \alpha)\)

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$$= \sum_{i=1}^{\infty} a_n \exp \left( \alpha \theta / \lambda_n \right) \times \exp \left( -\lambda_n s \right)$$ has the simple convergence-abscissa \( \sigma \), where (i) \( \theta \) is a real constant, (ii) \( \alpha \) is a constant determined later. Hence \( G_2(s) = \sum_{n \in \mathbb{N}} a_n \exp \left( -\lambda_n s \right) \) is simply convergent at least for \( \sigma > \sigma \). Now let us put

$$F(s; \theta, \alpha) = G_1(s; \theta, \alpha) + G_2(s),$$

which is evidently simply convergent at least for \( \sigma > \sigma \).

Denote by \( E(\theta, \alpha) \) the set of regular points of \( F(s; \theta, \alpha) \) on \( \sigma = \sigma \), which is clearly an open set. Then we can prove that

$$E(\theta_1, \alpha) \cap E(\theta_2, \alpha) = 0 \quad \text{for} \ \theta_1 \neq \theta_2.$$  

In fact, if there should exist a point \( \xi \) on \( \sigma = \sigma \) such that \( \xi \in E(\theta_1, \alpha) \cap E(\theta_2, \alpha) \neq 0 \), then \( F(s; \theta_1, \alpha) - F(s; \theta_2, \alpha) \) would be regular at \( s = \xi \).

On the other hand, since

$$F(s; \theta_1, \alpha) - F(s; \theta_2, \alpha)$$

$$= \sum_{i=1}^{\infty} a_n \left\{ \exp \left( \alpha \theta_1 / \lambda_n \right) - \exp \left( \alpha \theta_2 / \lambda_n \right) \right\} \exp \left( -\lambda_n s \right)$$

$$= \sum_{i=1}^{\infty} a_n \lambda_n \exp \left( -\lambda_n s \right),$$

taking account of (3), G. Valiron’s theorem, and Carlson-Landau’s theorem \([3, pp. 140–141]\), \( F(s; \theta_1, \alpha) - F(s; \theta_2, \alpha) \) has the simple convergence-abscissa \( \sigma \), and furthermore it has \( \sigma = \sigma \) as the natural boundary, which contradicts the regularity at \( s = \xi \). Hence, (5) holds.

If \( E(\theta, \alpha) \neq 0 \) should hold for all \( \theta, 0 \leq \theta \leq \gamma \) (\( \gamma \) a fixed constant), then, by (5), \( \{ F(s; \theta, \alpha) \} \) is at most of enumerable power, which contradicts the power of continuum of \( \{ F(s; \theta, \alpha) \} \). Hence, for at least one \( \theta' \), \( E(\theta', \alpha) = 0 \) holds. If we put \( \alpha = (-1)^{1/2}(-1) \), then (a) ((b)) is valid. q.e.d.

REFERENCES


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