A SET OF GENERATING FUNCTIONS FOR BESSEL POLYNOMIALS

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1. **Introduction.** The Bessel polynomials studied by Krall and Frink[1] are in essence hypergeometric functions which may be written as

$$f_n^{(c)}(x) = {}_2F_0(-n, c+n; -; x). \tag{1}$$

In terms of the notation in the paper cited in footnote 1,

$$y_n(x, a, b) = f_n^{(a-1)}(-x/b); \tag{2}$$

this paper will keep the $f_n^{(c)}(x)$ notation to avoid carrying the extra parameter b.

At least two generating functions are known for Bessel polynomials.[2] The purpose of this paper is to present another, namely the divergent[3] generating function

$$\begin{aligned} &{}_2F_0\left(\alpha, c-\alpha; -; \frac{t-(t^2-4xt)^{1/2}}{2}\right) \\ &\cdot {}_2F_0\left(\alpha, c-\alpha; -; \frac{t+(t^2-4xt)^{1/2}}{2}\right) \sim \sum_{n=0}^{\infty} \frac{(\alpha)_n(c-\alpha)_n f_n^{(c)}(x)t^n}{n!} \end{aligned} \tag{3}$$

where α is an arbitrary parameter. Thus (3) really gives a whole set of generating functions, one for each value of the parameter α.

Putting α equal to the nonpositive integer $-k$ $(k=0, 1, 2, \cdots)$ gives a finite series which is easily written as

$$\begin{aligned} &f_k^{(c)}\left(\frac{t-(t^2-4xt)^{1/2}}{2}\right) f_k^{(c)}\left(\frac{t+(t^2-4xt)^{1/2}}{2}\right) \\ &\qquad = \sum_{n=0}^{k} \frac{(-k)_n(c+k)_n f_n^{(c)}(x)t^n}{n!}. \end{aligned} \tag{4}$$

Received by the editors June 23, 1952.

[1] H. L. Krall and Orrin Frink, *A new class of orthogonal polynomials; the Bessel polynomials*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 100–115.

[2] See J. L. Burchnall, *The Bessel polynomials*, Canadian Journal of Mathematics vol. 3 (1951) pp. 62–68, and E. D. Rainville, *Generating functions for Bessel and related polynomials*, Canadian Journal of Mathematics vol. 5 (1953) pp. 104–106.

[3] Each ${}_2F_0$ in (3) diverges except for nonpositive integer parameters or for zero argument. Equation (3) means that the coefficient of each power of t on the left equals the coefficient of the corresponding power on the right. Fortunately convergence is not in general necessary in using generating functions.

It may be noted that the right side of (4) has the neat symbolic form

$$f_k^{(c)}(tf^{(c)}(x)). \tag{5}$$

2. **Proof of result (3).** First consider the straightforward Cauchy product

$$\begin{aligned} &{}_2F_0(\alpha, \beta; -; u)\,{}_2F_0(\alpha, \beta; -; v) \\ &\qquad \sim \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\alpha)_{n-k}(\beta)_{n-k}(\alpha)_k(\beta)_k u^{n-k}v^k}{(n-k)!k!}. \end{aligned} \tag{6}$$

The right-hand side becomes

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n u^n}{n!} \sum_{k=0}^{n} \frac{(\alpha)_k(\beta)_k(-n)_k(-v/u)^k}{(-\alpha-n+1)_k(-\beta-n+1)_k k!}$$

and it follows that

$$\begin{aligned} &{}_2F_0(\alpha, \beta; -; u)\,{}_2F_0(\alpha, \beta; -; v) \\ &\qquad \sim \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n u^n}{n!}\, {}_3F_2\left[\begin{matrix}\alpha, \beta, -n; \\ 1-\alpha-n, 1-\beta-n;\end{matrix} \; -v/u\right]. \end{aligned} \tag{7}$$

Using (7) on the left-hand side of (3) gives

$$\begin{aligned} &{}_2F_0\left(\alpha, c-\alpha; -; \frac{t-(t^2-4xt)^{1/2}}{2}\right) \\ &\cdot {}_2F_0\left(\alpha, c-\alpha; -; \frac{t+(t^2-4xt)^{1/2}}{2}\right) \\ &\qquad \sim \sum_{n=0}^{\infty} \frac{(\alpha)_n(c-\alpha)_n}{n!}\left(\frac{t-(t^2-4xt)^{1/2}}{2}\right)^n \\ &\cdot {}_3F_2\left[\begin{matrix}\alpha, c-\alpha, -n; \\ 1-\alpha-n, 1+\alpha-c-n;\end{matrix} \; \frac{-t-(t^2-4xt)^{1/2}}{t-(t^2-4xt)^{1/2}}\right]. \end{aligned} \tag{8}$$

Now the ${}_3F_2$ on the right-side of (8) may be converted by a formula due to Whipple,[4]

$$\begin{aligned} &{}_3F_2(-n, \beta, \gamma; -n-\beta+1, -n-\gamma+1; z) \\ &= (1-z)^n\, {}_3F_2\left[\begin{matrix}-n/2, (-n+1)/2, -n-\beta-\gamma+1; \\ -n-\beta+1, -n-\gamma+1;\end{matrix} \; \frac{-4z}{(1-z)^2}\right]. \end{aligned} \tag{9}$$

[4] F. J. W. Whipple, *Some transformations of generalized hypergeometric series*, Proc. London Math. Soc. (2) vol. 26 (1927) p. 267. This formula, with credit to Whipple, is also listed by Magnus and Oberhettinger, *Formulas and theorems for the special functions of mathematical physics*, Chelsea, 1949, p. 11.

This converts the right-hand side of (8) into

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n(c-\alpha)_n t^n}{n!} {}_3F_2\left[\begin{matrix} -n/2,\ (-n+1)/2,\ 1-n-c; \\ 1-n-\alpha,\ 1-n-c+\alpha; \end{matrix} \frac{4x}{t}\right] \tag{10}$$

or

$$\sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{(c)_n(\alpha)_{n-k}(c-\alpha)_{n-k} t^n(-x/t)^k}{(n-2k)!(c)_{n-k}k!}. \tag{11}$$

A shift of index changes (11) into

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(c)_{n+2k}(\alpha)_{n+k}(c-\alpha)_{n+k} t^{n+k}(-x)^k}{n!(c)_{n+k}k!} \tag{12}$$

and a second shift gives

$$\sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{(c)_{p+k}(\alpha)_p(c-\alpha)_p t^p(-x)^k}{(p-k)!(c)_p k!} \tag{13}$$

or

$$\sum_{p=0}^{\infty} \frac{(\alpha)_p(c-\alpha)_p t^p}{p!} {}_2F_0(-p,\ c+p;\ -;\ x). \tag{14}$$

Thus (8)–(14) yield

$$\begin{aligned} &{}_2F_0\left(\alpha,\ c-\alpha;\ -;\ \frac{t-(t^2-4xt)^{1/2}}{2}\right) \\ &\cdot {}_2F_0\left(\alpha,\ c-\alpha;\ -;\ \frac{t+(t^2-4xt)^{1/2}}{2}\right) \\ &\qquad \sim \sum_{n=0}^{\infty} \frac{(\alpha)_n(c-\alpha)_n t^n}{n!} {}_2F_0(-n,\ c+n;\ -;\ x) \end{aligned} \tag{15}$$

which is equivalent to (3) by use of (1).

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