CONTINUOUS WHICH ARE THE SUM OF A FINITE NUMBER OF INDECOMPOSABLE CONTINUA

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Swingle [7] has given the following definitions. (1) A continuum \( M \) is said to be the \textit{finished sum} of the continua of a collection \( G \) if \( G^* = M \) and no continuum of \( G \) is a subset of the sum of the others.\(^2\) (2) If \( n \) is a positive integer, the continuum \( M \) is said to be \textit{indecomposable under index} \( n \) if \( M \) is the finished sum of \( n \) continua and is not the finished sum of \( n+1 \) continua.

Swingle has shown [7, Theorem 2] that if \( n \) is a positive integer and the continuum \( M \) is indecomposable under index \( n \), then \( M \) is the finished sum of \( n \) indecomposable continua. The author has shown [2, Theorem 1] that if \( n = 2 \) and the continuum \( M \) is indecomposable under index \( n \), and \( G \) is a collection of \( n \) indecomposable continua whose finished sum is \( M \), then \( G \) is the only such collection. In the present paper, it is shown that for a compact continuum, this theorem holds for any positive integer \( n \). Also, there is given a necessary and sufficient condition that a compact continuum be indecomposable under index \( n \).

An indecomposable continuum can be described as a nondegenerate continuum which is indecomposable under index 1. If \( n = 1 \), then in order that a continuum \( M \) be indecomposable under index \( n \), it is necessary and sufficient that \( M \) contain \( n+2 \) points such that \( M \) is irreducible about any \( n+1 \) of them.\(^3\) Swingle [7] has shown that it is impossible, in a certain manner, to generalize this theorem. Theorem 3 of the present paper might be considered a generalization of the necessary condition of the above theorem. However, it is easily seen that the converse of Theorem 3 is not true.

Theorems 1–5 are proved on the basis of R. L. Moore's Axioms 0 and 1. Hence these theorems hold in any metric space.\(^4\)

**Theorem 1.** If \( n > 1 \) and the compact continuum \( M \) is the sum of \( n \) indecomposable continua \( M_1, M_2, \ldots, M_n \) such that, for each \( i (1 \leq i \leq n) \), a composant \(^6\) \( K_i \) of \( M_i \) does not intersect \( M_1 + M_2 + \cdots + M_{i-1} \)

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\(^1\) Numbers in brackets refer to the bibliography at the end of this paper.

\(^2\) The sum of the continua of \( G \) is denoted by \( G^* \).

\(^3\) For a proof of this theorem, see [4, Theorem IV].

\(^4\) Moore's axioms are stated in [5]. The first three parts of Axiom 1 are denoted by Axiom 1.

\(^6\) If \( P \) is a point of a continuum \( M \), the set of all points \( X \) such that \( P + X \) lies in a proper subcontinuum of \( M \) is called a composant of \( M \).
+ M_{i+1} + \cdots + M_n$, then $M$ is indecomposable under index $n$.

Proof. Suppose that there is a collection $G$ consisting of $n+1$ continua whose finished sum is $M$. No continuum of $G$ is a proper subset of one of the indecomposable continua $M_1, M_2, \ldots, M_n$. Hence, for each $i$ ($i \leq n$), if $K_i$ intersects a continuum $X$ of $G$, then $X$ contains $M_i$. Consequently, there exist $n$ continua of $G$ such that their sum is $M$. This is contrary to the supposition that $M$ is the finished sum of the continua of $G$. Since $M$ is the finished sum of the continua $M_1, M_2, \ldots, M_n$, then it is indecomposable under index $n$.

Theorem 2. If $n$ is a positive integer and the compact continuum $M$ is indecomposable under index $n$, then there is only one collection of indecomposable continua whose finished sum is $M$.

Proof. By [7, Theorem 2], there is a collection $G$ consisting of $n$ indecomposable continua $M_1, M_2, \ldots, M_n$ such that $M$ is their finished sum. By [3, Theorem 1], for each $i$ ($i \leq n$), some composant $K_i$ of $M_i$ does not intersect $(G - M_i)^*$. Suppose that there is a collection $G'$ of indecomposable continua such that $G' \neq G$ and $M$ is the finished sum of the continua of $G'$. Let $i$ be a positive integer not greater than $n$. Some continuum $X_i$ of $G'$ intersects $K_i$. Neither of the indecomposable continua $X_i$ and $M_i$ is a proper subset of the other. Since no proper subcontinuum of $M_i$ intersects both $K_i$ and $(G - M_i)^*$, then $X_i = M_i$. Hence $G' = G$.

Theorem 3. If $n > 1$ and the compact continuum $M$ is indecomposable under index $n$, then there is a subset $H$ of $M$ consisting of $2n$ points such that $M$ is irreducible about every subset of $H$ consisting of $2n - 1$ points.

Proof. Let $M_1, M_2, \ldots, M_n$ be $n$ indecomposable continua whose finished sum is $M$. For each $i$ ($i \leq n$), let $K_i$ be a composant of $M_i$ as described in the proof of Theorem 2. There exists a subset $H$ of $M$ such that for each $i$ ($i \leq n$), $H \cdot M_i$ consists of two points of $K_i$. The set $H$ satisfies the requirements of the conclusion of Theorem 3.

Theorem 4. If $n > 1$, $M$ is a compact continuum, $G$ is a collection consisting of $n$ indecomposable continua whose finished sum is $M$, and $H$ is a finite set of points about which $M$ is irreducible, then $M$ is indecomposable under index $n$.

Lemma 4.1. If the hypothesis of Theorem 4 is satisfied, $X$ is a continuum of $G$, and $T$ is a component of $(G - X)^*$, then some composant of $X$ does not intersect $T$. 
Proof of Lemma 4.1. Suppose that every composant of $X$ intersects $T$. Then there exists a finite collection $W$ of proper subcontinua of $X$ such that $W^* + (G - X)^*$ is connected. There exists a finite collection $Y$ of proper subcontinua of $X$ such that (1) every continuum of $Y$ intersects $(G - X)^*$ and (2) if $X$ intersects $H$, then $Y^*$ contains $X \cdot H$. Since $X$ is indecomposable and $M$ is the finished sum of the continua of $G$, then $Y^* + W^*$ does not contain $M - (G - X)^*$. Therefore, $W^* + Y^* + (G - X)^*$ is a proper subcontinuum of $M$ containing $H$. This is a contradiction since $M$ is irreducible about $H$.

Proof of Theorem 4. An inductive argument will be used. Suppose that Theorem 4 is not true. Let $k$ be the smallest positive integer $n$ such that if $M$ is a compact continuum satisfying the hypothesis of Theorem 4, then $M$ is not indecomposable under index $n$. By Theorem 1, there is a continuum $X$ of $G$ such that every composant of $X$ intersects $(G - X)^*$. By Lemma 4.1, $(G - X)^*$ is not connected. Therefore, $k > 2$. The set $(G - X)^*$ is the sum of a finite number of mutually exclusive continua. Let $T$ be one of these continua. Since $M$ is irreducible about $H$, then $T - T \cdot X$ contains a point of $H$. By Lemma 4.1, there is a composant of $X$ which does not intersect $T$. Let $P$ be a point of such a composant. The continuum $T + X$ is irreducible about the finite set $H \cdot T + P$. There is a positive integer $j$ less than $k$ such that $T + X$ is the finished sum of $j$ continua of $G$. Then $T + X$ is indecomposable under index $j$. By [3, Theorem 1], every continuum of $G$ which is a subset of $T + X$ contains a composant which does not intersect any other continuum of $G$ which is a subset of $T + X$. Therefore, every continuum of $G - X$ contains a composant which does not intersect any other continuum of $G$. Let $L$ be a collection consisting of $k - 1$ points such that if $Z$ is a continuum of $G - X$, then a point of $L$ belongs to a composant of $Z$ lying in $M - (G - Z)^*$. Since, by supposition, $M$ is not indecomposable under index $k$, then there is a collection $G'$ consisting of $k + 1$ continua whose finished sum is $M$. Since the set $L$ is contained in the sum of $k - 1$ continua of $G'$, then $(G - X)^*$ is contained in the sum of $k - 1$ continua of $G'$. Hence there exist two continua $X_1$ and $X_2$ of $G'$ such that each of them contains a point of $M - (G - X)^*$ which does not belong to any other continuum of $G'$. Let $R$ be a domain intersecting $X_1$ and not intersecting $(G' - X_1)^* + (G - X)^*$. Every composant of $X$ intersects $R$. Therefore, there exists a finite collection $W$ of proper sub-

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* This follows from the fact that every proper subcontinuum of an indecomposable continuum $M$ is a continuum of condensation of $M$ [4, Theorem II] and the fact that no indecomposable continuum is the sum of a finite number of its proper subcontinua [4, Theorem III].
continua of \( X \) such that \( X_1 + W* + (G-X)* \) is a continuum. Let \( Y \)
be a finite collection of continua as described in the proof of Lemma
4.1. Since \( X_1 + Y* + W* + (G-X)* \) is a subcontinuum of \( M \) contain-
ing \( H \), then \( X_1 + Y* + W* + (G-X)* = M \). Since \( X \) is indecomposable
and \( X_1 + (G-X)* \) contains \( X-(Y*+W*) \), then \( X_1 + (G-X)* \) contains
\( X \). This is impossible since \( X_1 + (G-X)* \) does not contain \( X_2 \). Thus
the supposition that Theorem 4 is not true has led to a contradiction.

Theorem 5. If \( n > 1 \), then in order that the compact continuum \( M \)
should be indecomposable under index \( n \), it is necessary and sufficient
that \( M \) should be the finished sum of \( n \) indecomposable continua and be
irreducible about some \( n \) points.\(^7\)

The necessity follows from [7, Theorem 2] and [3, Theorem 2].
The sufficiency follows from Theorem 4.

Theorem 6. If the compact continuum \( M \) in the plane is the finished
sum of two indecomposable continua \( H \) and \( K \) such that some compositant
of \( H \) does not intersect \( K \), then \( M \) is indecomposable under index two.

Lemma 6.1. If the hypothesis of Theorem 6 is satisfied and \( K_1 \) and \( K_2 \)
are mutually exclusive simple discs\(^8\) intersecting \( K \) but not \( H \), then there
do not exist four mutually exclusive continua \( W_1, W_2, W_3, \) and \( W_4 \)
such that, for each \( i \) \((i \leq 4)\), \( W_i \) belongs to \( K \), intersects \( H \), and is ir-
reducible from \( K_1 \) to \( K_2 \).

Proof of Lemma 6.1. Suppose that there do exist four such
continua. Let \( D \) denote the complementary domain of \( K_1+K_2 \).
Consider the case in which \( W_3+W_4 \) separates \( W_1 \) from \( W_2 \) in \( D \). Let
\( R_1 \) and \( R_2 \) be connected domains intersecting \( H \cdot W_1 \) and \( H \cdot W_2 \)
respectively and not intersecting \( K_1+K_2+W_3+W_4 \). There is a com-
positant \( L \) of \( H \) which intersects both \( R_1 \) and \( R_2 \) and lies in \( M-K \). Then
\( L \) intersects \( K_1+K_2+W_3+W_4 \). This is a contradiction since \( M-K \)
do not intersect \( K_1+K_2+W_3+W_4 \).

Proof of Theorem 6. Suppose, on the contrary, that \( M \) is the
finished sum of three continua \( M_1, M_2, \) and \( M_3 \). One of these three
continua intersects a compositant of \( H \) lying in \( M-K \). Suppose that
\( M_1 \) is such a continuum. Then it contains \( H \) and intersects each of

\(^7\) For an example showing that this theorem does not hold true without the condi-
tion that \( M \) be irreducible about some \( n \) points, see [1, p. 540]. Also, see [2, Example
1]. Sorgenfrey [6] has proved a theorem giving a necessary and sufficient condition
that a compact continuum be irreducible about some \( n \) points.

\(^8\) In the plane, a simple closed curve together with its interior is called a simple
disc.
the continua $M_2$ and $M_3$. Each of the continua $M_3$ and $M_1 + M_2$
contains a point of $K$ not belonging to the other of these two con-
tinua. Since the closure of $M - (M_1 + M_2)$ is a proper subset of the
indecomposable continuum $K$, then $M - (M_1 + M_2)$ is not connected.
Let $T_1$ and $T_2$ be two mutually separated sets whose sum is $M$
$- (M_1 + M_2)$. Let $K_1$ and $K_2$ be two mutually exclusive simple discs
whose interiors intersect $T_1$ and $T_2$ respectively such that $K_1$ and
$K_2$ do not intersect $T_2 + M_1 + M_2$ and $T_1 + M_1 + M_2$ respectively.
Since every composant of $K$ intersects both $K_1$ and $K_2$, there exist
six distinct composants of $K$ each of which contains a continuum ir-
reducible from $K_1$ to $K_2$. By Lemma 6.1, at most three of these inter-
sect $H$, and hence three do not. Denote three which do not by $W_1$,
$W_2$, and $W_3$. Let $D$ denote the complementary domain of $K_1 + K_2$.
There exist two of the continua $W_1$, $W_2$, and $W_3$ such that their sum
separates the other one from $H$ in $D$. Consider the case in which
$W_1 + W_3$ separates $W_2$ from $H$ in $D$. Let $I$ denote the complementary
domain of $K_1 + K_2 + W_1 + W_2$ which contains the connected set
$W_2 - W_2 \cdot (K_1 + K_2)$. Since one of the sets $K_1 \cdot W_1$ and $K_2 \cdot W_2$
belongs to $T_1$ and the other to $T_2$, then $I \cdot W_2$ contains a point of the continuum
$M_1 + M_2$. Since $H$ is a subset of $M_1 + M_2$ and does not intersect $I$,
then there is a continuum $Z$ belonging to $I \cdot (M_1 + M_2)$ and intersecting
both $W_2$ and $W_1 + W_3$. But this is impossible since $Z$ is a proper sub-
continuum of $K$ intersecting two composants of $K$. Thus the sup-
position that $M$ is the finished sum of three continua has led to a
contradiction.

**Theorem 7.** If the hypothesis of Theorem 6 is satisfied, then un-
countably many composants of $K$ lie in $M - H$.

This theorem follows from Theorem 6 and [3, Theorem 1].

**Remark.** Neither Theorem 6 nor Theorem 7 holds true in Euclidean
three-dimensional space. Let $H'$ be the point set obtained by translat-
ing the point set $H$ of [2, Example 1] one-half unit to the left. Let $H''$
be a point set obtained by revolving $H'$ through 90 degrees about
the vertical line whose equation is $x = 1/2$. Only one composant of $H''$
intersects $H$, but every composant of $H$ intersects $H''$. It follows from
[3, Theorem 1] and Theorem 2 that the continuum $H + H''$ is not
indecomposable under index two.

**Added in proof.** I have recently observed that Theorem 6 follows
from Theorem 1 and a lemma proved by N. E. Rutt [Some theorems
on triodic continua, Amer. J. Math. vol. 56 (1934) pp. 122–132
Lemma 1]. I regret that I was not aware of Rutt's lemma at the time
I prepared this paper.
BIBLIOGRAPHY


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