NOTE ON S-NUMBERS

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The transcendental number $\xi$ is said to be an S-number if there is a $\gamma > 0$ and a sequence of positive constants $\Gamma_1, \Gamma_2, \cdots$ such that for each polynomial $a_0 + \cdots + a_m x^m$ of arbitrary degree $m$, with rational integral coefficients such that $a = \max (|a_0|, \cdots, |a_m|) \geq 1$, the inequality

\[ |a_0 + a_1 \xi + \cdots + a_m \xi^m| \geq \Gamma_m a^{-\gamma m} \]

holds. It is well known (cf. [1, p. 63]) that if $\xi$ is an S-number, then $\gamma \geq 1/\sigma$, where

\[ \sigma = \begin{cases} 1 & \text{if } \xi \text{ is real,} \\ 2 & \text{if } \xi \text{ is not real,} \end{cases} \]

and Khintchine [2] has shown that the measure of all real S-numbers with $\gamma = 1$ is zero. On the other hand, Mahler [3] proved that almost all (real or complex) numbers are S-numbers. His proof shows that this statement is true with $\gamma = 4$, and he conjectured that for almost all real numbers one can take $\gamma = 1 + \varepsilon$, and that for almost all complex numbers one can take $\gamma = 1/2 + \varepsilon$, for arbitrary $\varepsilon > 0$.

Let $\gamma_r$ be the infimum of all numbers $\gamma'$ such that almost all real numbers are S-numbers with $\gamma \leq \gamma'$, and define $\gamma_c$ analogously in the complex case; thus Mahler's conjecture is that $\gamma_r = 1$, $\gamma_c = 1/2$. Koksma [4] showed that $\gamma_r \leq 3$ and $\gamma_c \leq 5/2$. Kubilyus [5] considered the special case $m = 2$, and showed that then (1) holds for almost all $\xi$ if $\gamma > 1/\sigma$; i.e., he proved the conjecture for $m = 2$. (It is well known for $m = 1$.) By combining a theorem due to Fel'dman [6] with Mahler's original argument, it is shown here that $\gamma_r \leq 2$, $\gamma_c \leq 3/2$.

Lemma 5 of Fel'dman's paper is as follows: Let $f(z) = a_0 + \cdots + a_m z^m$, where $a_0, \cdots, a_m$ are rational integers with $|a_i| \leq a$, let $\xi_1, \cdots, \xi_m$ be its zeros, which are supposed distinct, and let $\xi$ be an arbitrary complex number. If $\delta = \min_i (|\xi - \xi_i|)$, then

\[ |f(\xi)| \geq \delta e^{-2m^2 a^{-m}}. \]

First consider the real case. If $f$ satisfies the conditions of Fel'dman's lemma and

\[ |f(x)| \leq \mu a^{-2m-3}, \]

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then

$$\delta \leq \mu e^{m^2} a^{-m^2}.$$  

Thus we have the following analogue of Lemma 2 of [3]: The point set $M(f)$ on the real axis, in which the polynomial $f(x)$ satisfies the inequality (2), has total length at most

$$2m e^{m^2} a^{-m^2}.$$  

The remaining argument of [3] can be applied almost unchanged to this variant of Lemma 2, and the inequality $\gamma, \leq 2$ results. (This constant 2 is the coefficient of $m$ in (2).)

In the complex case, if

$$|f(z)| \leq e^{-(\pi^2+1)/8},$$

then

$$\delta \leq \mu e^{m^3} a^{-(m+1)/2},$$

and so $z$ must lie in a set composed of at most $m$ circular regions, of total area at most

$$\pi m e^{m^3} a^{m-3},$$

and Mahler's argument leads to the inequality $\gamma, \leq 3/2$.

**Bibliography**


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