SOME COUNTEREXAMPLES IN THE CLASSIFICATION OF OPEN RIEMANN SURFACES

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Recently Ahlfors and the author [1] constructed a Riemann surface of hyperbolic type which possessed no nonconstant harmonic functions with a finite Dirichlet integral. In the first section we explore some of the consequences of this example and construct a Riemann surface on which the spaces $HD$ and $HBD$ have dimension $n$. In the next section a bounded Riemann surface is exhibited which has no $HD$ functions on it which vanish on the relative boundary, while it has a nonconstant $HD$ whose normal derivative vanishes on the relative boundary. In the last section we use a refinement of the method in [1] to construct a Riemann surface admitting a nonconstant bounded harmonic function, but no nonconstant harmonic functions with a finite Dirichlet integral, thus demonstrating that the classes $O_{HB}$ and $O_{HD}$ are distinct.

1. Consider a sequence $\frac{1}{2} < r_1 < \cdots < r_n < \cdots < 1$ and the segments

$$\Delta_n: \begin{cases} r_n \leq r \leq r_{n+1}, \\ \theta = 2\pi h \cdot 2^{-n}, \end{cases} \quad 0 \leq h < 2^n.$$ 

We divide each $\Delta_n$ into $2^n$ subsegments $\Delta_n^{h,k}$ of equal logarithmic length and form a Riemann surface $W$ by identifying the left edge of $\Delta_n^{h,k}$ with the right edge of $\Delta_n^{h+k}$, where it is to be understood that $h+k$ is reduced to its remainder mod $2^n$. It was shown in [1] that $W$ has no nonconstant harmonic functions with a finite Dirichlet integral defined on it, provided

$$\sup r_n \leq 2^n \log \frac{1}{r_n} = \infty.$$ 

Let $V$ be the surface formed by removing the circle $K:r<1/2$ from $W$, and denote the circumference $r=1/2$ by $R$. We use $HN = HN(V)$ to denote the space of harmonic functions on $V$ which have a finite Dirichlet integral and whose normal derivative vanishes on $R$ and use $HO = HO(V)$ to denote the space of those functions which have a finite Dirichlet integral and which vanish on $R$.

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By Theorem 10 of [2] the space $HBD(W)$ of those harmonic functions on $W$ which have a finite Dirichlet integral and are bounded is isomorphic to

$$HO(V) + HO(K)$$

and to

$$HN(V) + HN(K).$$

Since $W$ has a Green's function, we include the constants in $HBD$ by convention. Hence $HBD$ consists of the constants alone and is thus one-dimensional. The compactness of $K$ implies that $HO(K)$ and $HN(K)$ contain only the function zero. Remembering that $HBD$ is dense in $HD$ in the sense of the Dirichlet metric [2], we also have $HD$ one-dimensional and hence:

**The spaces $HO(V)$ and $HN(V)$ have dimension one. Consequently $HO(V)$ consists of multiples of log $(r/2)$ while $HN(V)$ consists of constants.**

If we reflect $V$ in the circle $R$, we obtain a surface $W_2$ on which the space $HD$ has dimension 2. Thus if $u$ is harmonic and has a finite Dirichlet integral we must have

$$u = C_1 + C_2 \log r.$$ 

Similarly, if we start from the complex sphere from which $n+1$ circular disks have been removed and attach a replica of $V$ in place of each disk, we obtain a surface $W_n$ with the property that $HBD(W_n)$ is $n$-dimensional. Thus:

**For each integer $n$ there exists a Riemann surface $W_n$ on which the spaces $HD(W_n)$ and $HBD(W_n)$ are $n$-dimensional.**

**2.** Consider a sequence $0 < \eta_1 < \cdots < \eta_n < \cdots < 1$, set $\eta_n = -\eta_n$, and form the segments

$$\Delta_n^k: \begin{cases} 
\eta_n \leq y \leq \eta_{n+1}, \\
x = (k + 1)2^{-n+1}, \\
0 \leq k \leq 2^n - 1.
\end{cases}$$

We divide each $\Delta_n^k$ into $2^n$ subsegments $\Delta_n^{k,\alpha}$ of equal lengths and construct a bounded Riemann surface $V'$ from the interior of the rectangle $|y| \leq 1, 0 \leq x \leq 1$, by identifying the left edge of $\Delta_n^{k,\alpha}$ with the right edge of $\Delta_n^{k+1,\alpha}$, where here it is understood that $k+k$ is reduced to its remainder mod $2^n - 1$. The relative boundary $R$ of $V'$ consists of the segments $x=0$ and $x=1$.

Let $u$ be a harmonic function defined on $V'$ which has a finite Dirichlet integral and vanishes on $R$. Then, given $\delta > 0$, there is a set
of measure greater than $1 - \delta$ such that

$$\int_{-1}^{1} \left[ \frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2 \right] dy < M \delta$$

for $x \in E_\delta$. Now

$$|u(x, y_2) - u(x, y_1)|^2 = \left( \int_{y_1}^{y_2} \frac{\partial u}{\partial y} dy \right)^2 \leq (y_2 - y_1) M \delta$$

by the Schwarz inequality, and hence there is a function $l(x)$ such that

$$\lim_{\gamma \to 1} u(x, y) = l(x)$$

uniformly for $x \in E_\delta$. Thus for $\varepsilon > 0$, we may choose $N$ so that $l(x)^2 \leq u(x, y)^2 + \varepsilon$ for all $x \in E_\delta$ and all $y \geq \eta_N$.

Since the right edge of the interval $\Delta^0_n$ is identified with the left edge of $\Delta^0_n$, we have

$$u(x, y) = \int_0^{2^n} \frac{\partial u}{\partial x} dx + \int_{(h+1)2^n}^{2^n} \frac{\partial u}{\partial x} dx$$

for $(h+1)2^{-n} \leq x \leq (h+2)2^{-n}$ and for $y$ in the projection of $\Delta^0_n$. By the Schwarz inequality

$$u(x, y)^2 \leq 2^{-n} \int_0^{2^n} \frac{\partial u}{\partial x}^2 dx + 2^{-n} \int_{(h+1)2^n}^{2^n} \frac{\partial u}{\partial x}^2 dx,$$

whence

$$u^2 \leq 2^{-n+1} \int_0^{2^{-n+1}} \left[ \frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2 \right] dx.$$

For $n \geq N$ and $x$ in $I_n$, the intersection of $E_\delta$ with the interval $(h+1)2^{-n} \leq x \leq (h+2)2^{-n}$, we have also

$$l(x)^2 \leq 2^{-n+1} \int_0^{2^{-n+1}} \left[ \frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2 \right] dx + \varepsilon.$$ 

Integrating with respect to $y$ in the projection $P^\delta$ of $\Delta^0_n$ gives

$$2^{-n}(\eta_{n+1} - \eta_n)l(x)^2 \leq 2^{-n+1} \int_{P^\delta} \int_0^{1} \left[ \frac{\partial u}{\partial x}^2 + \frac{\partial u}{\partial y}^2 \right] dxdy + 2^{-n}(\eta_{n+1} - \eta_n).$$
Integrating with respect to $h$ in $I_n$ gives

$$2^{-n}(\eta_{n+1} - \eta_n) \int_{I_n} l(x)^2dx \leq 2^{-2n+1} \int_{I_h} \int_0^1 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dxdy$$

$$+ \epsilon 2^{-2n}(\eta_{n+1} - \eta_n).$$

Thus (if we put $l = 0$ outside $E_\delta$)

$$(\eta_{n+1} - \eta_n) \int_{z^{-N}}^{1-2^{-N}} l^2dx \leq \sum_{k=1}^{2n-2} (\eta_{n+1} - \eta_n) \int_{I_h} l^2dx$$

$$\leq 2^{-n+1} \int_{\eta_n}^{\eta_{n+1}} \int_0^1 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dxdy$$

$$+ \epsilon (\eta_{n+1} - \eta_n).$$

Therefore

$$(\eta_{n+1} - \eta_n) \int_{z^{-N}}^{1-2^{-N}} l^2dx \leq 2^{-n+1}D(u) + \epsilon (\eta_{n+1} - \eta_n).$$

Summing for $n \geq N$ gives

$$(1 - \eta_N) \int_{z^{-N}}^{1-2^{-N}} l^2dx \leq 2^{-N+1}D(u) + \epsilon (1 - \eta_N).$$

If we let $\eta_n$ converge to one so slowly that

$$\text{sup } 2^n(1 - \eta_n) = \infty,$$

then

$$\int_0^1 l^2dx < \epsilon,$$

and since the left-hand side is independent of $n$,

$$\int_0^1 l^2dx = 0,$$

hence $l = 0$ almost everywhere in $E_\delta$. Since $\delta$ is arbitrary we must have

$$\lim_{\nu \to -1} u(x, y) = 0$$

for almost all $x$ in $[0, 1]$. Similarly

$$\lim_{\nu \to -1} u(x, y) = 0$$

for almost all $x$ in $[0, 1]$. 
For almost all $x$ we then have
\[ u(x, y) = \int_{-1}^{y} \frac{\partial u}{\partial y} \, dy \]
and by the Schwarz inequality
\[ u(x, y)^2 \leq 2 \int_{-1}^{1} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] \, dy = \mu(x). \]
But $\mu$ is summable since
\[ \int_{0}^{1} \mu(x) \, dx = 2D(u). \]
Hence
\[ m(y) = \int_{0}^{1} u^2 \, dx \]
is continuous and $m(1) = m(-1) = 0$. Moreover
\[
m'(y_2) - m'(y_1) = 2 \int_{0}^{1} u \frac{\partial u}{\partial y} \, dx \bigg|_{y=y_2} - 2 \int_{0}^{1} u \frac{\partial u}{\partial y} \, dx \bigg|_{y=y_1} = 2 \int_{y_1}^{y_2} \int_{0}^{1} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] \, dxdy,
\]
since $u$ is a harmonic function which vanishes for $x=0$ and $x=1$. Thus $m'(y)$ is increasing whence $m(y)$ is convex. But $m(y)$ must vanish identically in the interval $-1 \leq y \leq 1$, since it is non-negative, convex, and vanishes at the end points. Hence $u=0$.

The function $u=y$ is single-valued on $V'$ and has a finite Dirichlet integral, while $\partial u/\partial n=0$ on $R$. Thus we have the following result.

**On the bounded Riemann surface $V'$ the class HO is empty, while there is a nonconstant harmonic function with a finite Dirichlet integral whose normal derivative vanishes on the relative boundary.**

3. We form a Riemann surface $W'$ in the strip $|y| \leq 1$ by putting in replicas of $V'$ in each rectangle $n \leq x \leq n+1$. Let $u$ be a bounded harmonic function with a finite Dirichlet integral defined on $W'$. Then a slight modification in the argument of the preceding section shows that
\[ \lim_{v \to 1} u(x, y) = c \]
for almost all $x$, and
\[
\lim_{y \to -1} u(x, y) = c'
\]
for almost all \(x\).

Without loss of generality we may take \(c' = 0\). Then

\[
u(x, y) = \int_{-1}^{y} \frac{\partial u}{\partial y} \, dy
\]

and by the Schwarz inequality

\[
u(x, y)^2 \leq 2 \int_{-1}^{y} \left( \frac{\partial u}{\partial y} \right)^2 \, dy \leq 2 \int_{-1}^{1} \left( \frac{\partial u}{\partial y} \right)^2 \, dy = \mu(x).
\]

Since

\[
\int_{-\infty}^{\infty} \mu(x) \leq D(u) < \infty,
\]

\(\mu\) is summable and so

\[
m(y) = \int_{-\infty}^{\infty} u(x, y)^2 \, dx
\]
exists and is continuous for \(-1 \leq y \leq 1\). Also we have \(c^2 \leq \mu(x)\), whence \(c\) must be zero in order for \(\mu\) to be summable on \((-\infty, \infty)\). Hence, \(m(y_1) = m(y_2) = 0\). By the Schwarz inequality we have

\[
\left( \iint \left| \frac{\partial u}{\partial y} \right| \, dxdy \right)^2 \leq \iint u^2 \, dxdy \iint \left( \frac{\partial u}{\partial y} \right)^2 \, dxdy
\]

\[
\leq D(u)^2 < \infty.
\]

Thus by the Fubini theorem

\[
m_1(y) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial u}{\partial y} \, dx
\]
exists for almost all \(y\) and

\[
m(y_2) - m(y_1) = \frac{1}{2} \int_{-\infty}^{\infty} \int_{y_1}^{y_2} \frac{\partial u}{\partial y} \, dy \, dx
\]

\[
= \frac{1}{2} \int_{y_1}^{y_2} \int_{-\infty}^{\infty} \frac{\partial u}{\partial y} \, dxdy
\]

\[
= \int_{y_1}^{y_2} m_1(y) \, dy,
\]
whence $m$ is absolutely continuous and

$$m' = m_1 \text{ a.e.}$$

Let $y_1$ and $y_2$ be two values of $y$ for which $m_1$ exists, and take $N$ so large that $y_1 < \eta_N$, $y_2 < \eta_N$. Since

$$\int \int \left[ \left( \frac{\partial m}{\partial x} \right)^2 + \left( \frac{\partial m}{\partial y} \right)^2 \right] dxdy < \infty,$$

we can find an arbitrarily large $x_0$ such that the fractional part of $x_0$ lies between zero and $2^{-N}$ and such that

$$\int \left[ \left( \frac{\partial m}{\partial x} \right)^2 + \left( \frac{\partial m}{\partial y} \right)^2 \right] dy < \frac{\epsilon^2}{2}$$

on the two segments $x = \pm x_0$. On these segments

$$\left( \int \left| \frac{\partial m}{\partial x} \right| dy \right)^2 \leq 2 \int \left( \frac{\partial m}{\partial x} \right)^2 dy < \epsilon^2$$

and so

$$\int \left| \frac{\partial m}{\partial x} \right| dy < \epsilon.$$

We also take $x_0$ so large that

$$\int \int_{|x| > x_0} \left[ \left( \frac{\partial m}{\partial x} \right)^2 + \left( \frac{\partial m}{\partial y} \right)^2 \right] dxdy < \epsilon$$

and

$$\int_{|x| > x_0} \left| \frac{\partial u}{\partial y} \right| dx <$$

for $y = y_1, y_2$.

Since the fractional part of $x_0$ lies between zero and $2^{-N}$, the region $\Omega$ on $W'$ for which $|x| < x_0$ and $y_1 \leq y \leq y_2$ has as its boundary the segments $|x| < x_0$, $y = y_1, y_2$ and the segments $y_1 < y < y_2$, $|x| = x_0$. Thus by Green's theorem

$$D_0(u) = \int u \frac{\partial u}{\partial n} = \int_{x = -x_0} u \frac{\partial u}{\partial x} dy - \int_{x = x_0} u \frac{\partial u}{\partial x} dy + \int_{y = y_2} u \frac{\partial u}{\partial x} dy - \int_{y = y_1} u \frac{\partial u}{\partial x} dy$$

where the ranges of integration are $y_1 \leq y \leq y_2$, $-x_0 \leq x \leq x_0$. 

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Thus

\[ |D_0(u) - m_1(y_2) + m_1(y_2)| < 2M\epsilon + 2\epsilon \]

where \( M \) is a bound for \( |u| \). Consequently

\[ \left| \int_{y_1}^{y_2} \int_{-\infty}^{\infty} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dxdy - m_1(y_2) + m_1(y_1) \right| < (2M + 3)\epsilon. \]

The left-hand side must be zero since it is independent of \( \epsilon \), and so

\[ m_1(y_2) - m_1(y_1) = \int_{y_1}^{y_2} \int_{-\infty}^{\infty} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right] dx dy \geq 0. \]

From this and the fact that

\[ m(y) = \int_{-1}^{y} m_1(y)dy \]

we conclude that \( m(y) \) is convex. As a result \( m(y) = 0 \) in the interval \(-1 \leq y \leq 1\), since it is a non-negative convex function vanishing at the ends of the interval. This implies \( u = 0 \), and because of the identity \( \Omega_D = \Omega_{BB} \) we have the following proposition:

The Riemann surface \( W' \) has no nonconstant harmonic functions on it with a finite Dirichlet integral, while the function \( u = y \) is a harmonic function which is defined, single-valued, and bounded on \( W' \).

**Bibliography**
