A NOTE ON THE SCHOLZ-BRAUER PROBLEM IN ADDITION CHAINS

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1. Introduction. An addition chain for the positive integer \( n \) is a sequence of integers \( a_0 = 1 < a_1 < a_2 < \cdots < a_r = n \) where, for each \( i > 0 \), \( a_i = a_j + a_k \), for some \( j, k \geq i \) (\( j = k \) is permitted). For example, \( 1, 2, 4, 8, 10; 1, 2, 3, 6, 9, 10; 1, 2, 4, 6, 10 \) are three addition chains for \( n = 10 \). By \( l(n) \) one means the smallest \( r \) for which there is an addition chain for \( n \). One can easily verify that \( l(1) = 0, l(2) = 1, l(3) = l(4) = 2, l(5) = l(6) = l(8) = 3, l(7) = l(9) = l(10) = l(12) = l(16) = 4 \) and that \( l(2^n) = n \).

A. Scholz [2] published the following as problems:

(1) \( m + 1 \leq l(n) \leq 2m \) for \( 2^m + 1 \leq n \leq 2^{m+1}, m \geq 1 \),
(2) \( l(ab) \leq l(a) + l(b) \),
(3) \( l(2^s - 1) \leq q + l(q) - 1 \).

A. T. Brauer [1] established (1) and (2) and also improved another inequality suggested by Scholz. When considering (3) Brauer showed that (3) holds if the chains admitted in determining \( l(q) \) are restricted. So far as the author can discover, the original problem (3) of Scholz has not been solved. In this note we establish (3) for some values of \( q \) by a method that may extend to an arbitrary \( q \).

2. The case \( q = 2^s(2^n+1) \).

**Lemma 1.** \( l(2^s+1) = s+1, s \geq 0 \).

**Proof.** Clearly, \( l(2^s+1) \leq s+1 \) since

\[ 1, 2, 2^2, 2^3, \cdots, 2^s, 2^s + 1 \]

is an addition chain for \( 2^s+1 \). Also,

\[ 2^s - 1 < 2^s + 1 < 2^{s+1} \]

hence, by (1), \( l(2^s+1) \geq s+1 \) and the lemma is proved.

**Lemma 2.** \( l(2^s-1) \leq q + l(q) - 1 = 2^s + s - 1 \) if \( q = 2^s, s \geq 0 \).

**Proof.** This is shown by induction. The inequality holds for \( s = 0 \); suppose it holds for \( s = r - 1 \). Then setting \( m = 2^{-1} \),

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\[ l(2^{2^m} - 1) = l[(2^m - 1)(2^m + 1)] \leq l(2^m - 1) + l(2^m + 1) \]
\[ \leq 2^r + r - 1 \]

by Lemma 1 and the induction hypothesis. This completes the proof.

**Lemma 3.** \( l(2^{m+1} - 1) \leq q + l(q+1) = 2^s + s + 1 \) if \( q = 2^s, s \geq 0 \).

**Proof.** \( l(2^{m+1} - 1) \leq l(2^{m+1} - 2) + 1 \leq l(2) + l(2^g - 1) + 1 \leq 2^r + s + 1 \) by Lemma 2.

**Lemma 4.** \( l(2^{m}(2^s+1)) = n + s + 1, s, n \geq 0. \)

**Proof.** (a) \( 2^{s+t} - 1 < 2^{s}(2^s+1) < 2^{s+t+1} \) hence \( l(2^{m}(2^s+1)) \geq n + s + 1 \) by (1).

(b) \( l(2^{m}(2^s+1)) \leq l(2^s) + l(2^s+1) = n + s + 1 \) by Lemma 1 and property (2).

**Theorem 1.** \( l(2^{m} - 1) \leq q + l(q) - 1 = 2^{s}(2^s+1) + n + s \) if \( q = 2^{2^s+1}, s, n \geq 0. \)

**Proof.** This proof will be by induction on \( s \). We have seen in Lemma 3 that the theorem is true if \( s = 0 \). Assume, for the induction proof, that it holds for \( s = r - 1 \) and all \( n \geq 0 \). If \( m = 2^{r-1}(2^s + 1), \)
\[ l(2^{2^m} - 1) \leq l[(2^m - 1)(2^m + 1)] \]
\[ \leq 2^{r-1}(2^s + 1) + l(2^{r-1}(2^s + 1)) + 2^{r-1}(2^s + 1) \]

by the inductive hypothesis and Lemma 1. The proof is completed by the use of Lemma 4.

3. **Comments and questions.** Since any positive integer can be written
\[ 2^{a_1} + 2^{a_2} + 2^{a_3} + \cdots + 2^{a_n}, \text{ where } a_1 > a_2 > \cdots > a_n \geq 0, \]
one might expect to establish (3) by the ideas of this note (numbers for which \( n = 2 \) having now been disposed of). However, the author has been unable to carry this through.

Other inequalities involving \( l(p) \) would be of interest. One can easily show that \( l(a + b) \leq l(a) + l(b) \) if \( a, b > 1. \) Does the inequality \( l(p) < l(2^p) \) hold for all \( p > 0? \) Let \( S(n) \) denote the number of solutions of the equation \( l(x) = n. \) Is it true that \( S(n) < S(n+1) \) for all \( n > 0? \)

**Bibliography**