MINIMIZING OPERATORS ON SUBREGIONS

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For constructing harmonic functions with a prescribed local behavior, an operator method on arbitrary Riemann surfaces was recently introduced by the author [1]. We showed the existence of a normal linear operator minimizing the Dirichlet integral and referred to other operators to be given later. In the present paper, a general class of minimizing operators will be introduced, including the above operator as a special case. In the existence proof, use will be made of the extremal method presented in [2].

Let \( R \) be an arbitrary Riemann surface and \( G \) a subregion, compact or not, of finite or infinite genus, relatively bounded by a finite set \( \alpha \) of closed analytic Jordan curves. Let \( v \) be a real single-valued function on \( \alpha \), harmonic in an open set containing \( \alpha \). A normal linear operator \( L \) in \( G \) is defined [1] as follows. With every \( v \) on \( \alpha \) is associated, by \( L \), a unique single-valued harmonic function \( Lv \) on \( G \) which satisfies the following conditions:

1. \( Lv = v \) on \( \alpha \),
2. \( \min_{\alpha} v \leq Lv \leq \max_{\alpha} v \) on \( G \),
3. \( \int_{\alpha} dLv = 0 \),
4. \( L(c_1v_1 + c_2v_2) = c_1Lv_1 + c_2Lv_2 \).

Here \( Lv \) is the harmonic conjugate function of \( Lv \).

Denote by \( \{u\} \) the class of single-valued harmonic functions \( u \) in \( G \) with

5. \( u = v \) on \( \alpha \), \( \int_{\alpha} d\bar{u} = 0 \).

Let \( \beta \) be the ideal boundary of \( G \), that is, the common part of the boundaries of \( R \) and \( G \). If \( G \) is noncompact (\( \beta \) is not empty), we form an exhaustion \( \{G_n\} \) of \( G \) by domains \( G_n \), bounded by \( \alpha \) and a finite set \( \beta_n \) of closed analytic Jordan curves. The boundary integral \( \int_{\beta_n} u \, d\bar{u} \) is defined as the limit of integrals taken along the curves \( \beta_n \).

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These integrals increase monotonically with \( n \), for the Dirichlet integral of \( u \) over \( G_{n+1} - G_n \) is non-negative. If \( G \) is compact (\( \beta \) is empty), the boundary integral is understood to vanish for all \( u \). Let \( \lambda \) be a real parameter ranging in the interval \((- 1, 1)\).

**Theorem 1.** There is a uniquely determined function \( u_\lambda \) in \( G \) which minimizes the value of the functional

\[
m_\lambda(u) = \int_\beta u \, d\alpha + \lambda \int_\alpha u \, d\alpha
\]

among all functions of the class \( \{ u \} \). The function \( u_\lambda \) is associated with \( v \) by a normal linear operator \( L_\lambda \),

\[
u_\lambda = L_\lambda v.
\]

**Proof.** If \( G \) is compact, \( \{ u \} \) reduces to one single function and the theorem is trivial. In the sequel we assume that \( G \) is not compact. Suppose first that \( \beta \) consists of a finite number of closed analytic Jordan curves. Let \( u_1 \) and \( u_{-1} \) be the functions of class \( \{ u \} \) determined by

\[
u_1 = k = \text{const. on } \beta,
\]

\[
\partial u_{-1}/\partial n = 0 \text{ on } \beta
\]

where \( \partial/\partial n \) is the normal derivative. Write

\[
u_\lambda = \frac{1 + \lambda}{2} u_1 + \frac{1 - \lambda}{2} u_{-1},
\]

and set \( u - u_\lambda = h \). By \( h = 0 \) on \( \alpha \), we have

\[
m_\lambda(u) = \int_\beta u_\lambda \, d\alpha + \lambda \int_\alpha u_\lambda \, d\alpha + \int_\beta -a h \, d\alpha
\]

\[+ \int_\beta u_\lambda \, d\alpha + \lambda \int_\alpha u_\lambda \, d\alpha + \int_\beta -a h \, d\alpha,
\]

In view of the Green's formula

\[
\int_\beta -a h \, d\alpha = \int_\beta -a u_\lambda \, d\alpha,
\]

the sum of the three latter integrals may be written

\[
2 \int_\beta u_\lambda \, d\alpha + (\lambda - 1) \int_\alpha u_\lambda \, d\alpha.
\]
Substituting (10) in this and making use of (5), this reduces further to
\[
(1 - \lambda) \int_{\beta_a} u_{-1} d\bar{h} = (1 - \lambda) \int_{\beta_a} h d\bar{u}_{-1} = 0.
\]
Consequently,
\[
(11) \quad m_\lambda(u) = m_\lambda(u_\lambda) + D(u - u_\lambda),
\]
which shows that \(m_\lambda(u)\) is minimized by \(u_\lambda\). By (8) and (9), the functions \(u_1\) and \(u_{-1}\) are associated with \(v\) by a normal linear operator. The same is, therefore, true for \(u_\lambda\). This proves the theorem for the special \(\beta\) under consideration.

Now let \(\beta\) be arbitrary. Denote by \(u_{\lambda_n}\) the harmonic function in \(G_n\) which minimizes the value of
\[
m_{\lambda_n}(u) = \int_{\beta_n} u d\bar{u} + \lambda \int_{\alpha} u d\bar{u}
\]
among functions of the class \(\{u\}\) in \(G_n\). By (2), the functions \(u_{\lambda_n}\) are uniformly bounded, and a subsequence, say again \(\{u_{\lambda_n}\}\), converges uniformly in every closed subdomain of \(G\) towards a harmonic function \(u_\lambda\) on \(G\) with \(u_\lambda = v\) on \(\alpha\). In view of the harmonic boundary values and Schwarz's reflexion principle, the convergence is uniform even in a domain slightly extended across \(\alpha\). This implies that \(\text{grad } u_{\lambda_n}\) converges uniformly on \(\alpha\).

Since \(\int_{\alpha} u_{\lambda_n} d\bar{u}_{\lambda_n}\) increases with \(n(\leq m)\), it follows from the minimum property of \(u_{\lambda_n}\) that
\[
m_{\lambda_n}(u_{\lambda_n}) \leq m_{\lambda}(u_{\lambda_{n+1}}(u_{\lambda_{n+1}})).
\]
Similarly, for \(u\) in \(G\),
\[
m_{\lambda_n}(u_{\lambda_n}) \leq m_\lambda(u).
\]
As this holds for every \(n\) and every \(u\) in \(G\), we have
\[
\lim_{n \to \infty} m_{\lambda_n}(u_{\lambda_n}) \leq \inf m_\lambda(u) \leq m_\lambda(u_\lambda).
\]
Since, on the other hand,
\[
m_\lambda(u_\lambda) = \lim m_{\lambda_n}(u_\lambda) = \lim \lim m_{\lambda_n}(u_{\lambda_m}) \leq \lim m_{\lambda_m}(u_{\lambda_m}),
\]
it follows that
\[
(12) \quad m_\lambda(u_\lambda) = \min m_\lambda(u) = \lim m_{\lambda_n}(u_{\lambda_n}).
\]
This minimum property implies that, for a real \(\epsilon\),
\[ m_\lambda(u_\lambda + \epsilon h) = m_\lambda(u_\lambda) + \epsilon \left[ \int_\beta (u_\lambda d\bar{h} + h d\bar{u}_\lambda) + \lambda \int_\alpha (u_\lambda d\bar{h} + h d\bar{u}_\lambda) \right] + \epsilon^2 D(h). \]

The expression in brackets vanishes, since otherwise, for sufficiently small \(|\epsilon|\), the deviation of \(m_\lambda(u_\lambda + \epsilon h)\) from \(m_\lambda(u_\lambda)\) would change its sign with \(\epsilon\), contrary to the minimum property of \(u_\lambda\). For \(\epsilon = 1\), it follows that

\[ m_\lambda(u) = m_\lambda(u_\lambda) + D(u - u_\lambda). \]

This guarantees the uniqueness of \(u_\lambda\). In fact, let \(u'\) and \(u''\) be two minimizing functions. Then

\[ m_\lambda(u'') = m_\lambda(u') = m_\lambda(u'') + D(u' - u'') \]

which implies \(u' - u'' = \text{const.} = 0\). In particular, the sequence \(u_n\), not only a subsequence, converges.

Since \(u_n = L_n v\) satisfies the conditions (1)-(4) in \(G_n\), it follows from the uniform convergence that \(L_n\), defined by

\[ u_n = L_n v, \]

is a normal linear operator for \(G\). This completes the proof of Theorem 1.

We consider now the subclass \(\{u^0\}\) of \(\{u\}\), defined by the restriction \(\int d\bar{u}^0 = 0\) along all dividing cycles.

**Theorem 2.** There is a normal linear operator \(L^0_\lambda\) associating with \(v\) on \(\alpha\) a unique harmonic function

\[ \theta \]

\[ u_\lambda = L^0_\lambda v \]

on \(G\) which minimizes the value

\[ m_\lambda(u^0) = \int_\beta u^0 d\bar{u}^0 + \lambda \int_\alpha u^0 d\bar{u}^0 \]

among all functions of the class \(\{u^0\}\).

**Proof.** In the proof of Theorem 1, replace \(u_1\) by \(u_1^0 \subseteq \{u^0\}\), defined by

\[ u_1^0 = k_{n_1} = \text{const. on } \beta_{n_1}, \]

where \(\beta_{n_1}\) are the closed curves constituting \(\beta_n\). Write \(u_{-1}^0 = u_{-1}\) and replace \(u_\lambda\) by \(u^0_\lambda\), respectively. Then nothing in the previous proof
will be changed if the exhaustion \( \{ G_\alpha \} \) is (as is always possible) chosen so that each \( \beta_\alpha \) is a dividing cycle.

We now apply the operators introduced above to existence problems on the Riemann surface \( R \), on which the subregion \( G \) was considered. In \( R - \alpha \), let \( s \) be a single-valued real function, harmonic near \( \alpha \), both branches of which can be continued harmonically across \( \alpha \). Let \( L \) be a normal linear operator in \( R - \alpha \). The following theorem was proved in \([1]\). If \( \int d\bar{s} \) vanishes, when extended along both edges of \( \alpha \) for respective branches of \( s \), then, and only then, there exists on the whole surface \( R \) a function \( p \), harmonic on \( \alpha \) and such that \( p - s = L(p - s) \) in each of the disjoint regions constituting \( R - \alpha \). For \( L = L_\lambda \) (or \( L_0^* \)) this gives, in particular:

**Theorem 3.** On an arbitrary Riemann surface \( R \), let \( D \) be a compact region, bounded by a finite set \( \alpha \) of closed analytic Jordan curves. In \( D \), let \( s \) be a single-valued real function, harmonic on \( \alpha \). The condition

\[
\int_a d\bar{s} = 0
\]

is necessary and sufficient for the existence of a single-valued function \( p_\lambda \) (or \( p_0^* \)) on \( R \) such that

1. \( p_\lambda - s \) is harmonic on \( \overline{D} \),
2. \( p_\lambda \) is harmonic on \( R - D \),
3. the value of the functional

\[
m_\lambda(u) = \int_\beta ud\bar{u} + \lambda \int_\alpha ud\bar{u}
\]

is minimized by \( u = p_\lambda \) among all functions of the class \( \{ u \} \) (or \( \{ u^0 \} \)) in \( R - D \) with the boundary values \( p_\lambda \) on \( \alpha \).

The proof is furnished by the theorem quoted above, selecting \( s \equiv 0 \) in \( R - \overline{D} \).

Note that, for \( \lambda = -1 \), the operator \( L_\lambda \) is the special operator introduced in \([1]\) (denoted there by \( L_0 \)) which minimizes the Dirichlet integral. For \( \lambda = 1 \), \( L_\lambda \) minimizes \( \int_{\beta - \alpha} ud\bar{u} \), furnishing the function of Lemma 1 in \([4]\). For \( \lambda = 0 \), \( \int_{\beta} ud\bar{u} \) is minimized by \( L_\lambda \), the mean of the two above operators. Necessary and sufficient conditions, given in \([1]\) for the existence of certain harmonic and analytic functions, are valid in terms of any of the operators \( L_\lambda \).

The functions \( p_0^* - 1 \) and \( p_0^* \), corresponding to the operators \( L_0^* - 1 \) and \( L_0^* \) and to \( s = \text{Re} \left( \frac{1}{z} \right) \), are the real parts of functions mapping a planar surface onto the horizontal or vertical slit domains, respec-
tively. The functions $f^0$ have application to related mapping problems.

A survey of the linear operator method and the extremal method, to which this investigation is related, was given in [3].

References


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