A CLASS OF SUPER-ADDITIVE FUNCTIONS

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It is well known that the torsional rigidity is superadditive\(^1\) [1]. The torsional rigidity can be defined [2] as the integral of the stress function \(u(p; D)\) over the domain \(D\), which is also super-additive (in the sense of Definition 3). In this paper we show that the stress function (suitably extended in definition), and hence the torsional rigidity, is super-additive in a more restrictive sense. Specifically we prove that if \(D_1, D_2, \ldots, D_n\) are domains contained in a domain \(D\), then

\[
u(p; D) \geq \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq m_1 < m_2 < \cdots < m_k \leq n} u(p; \bigcap D_{m_i}).
\]

1. Definitions.

**Definition 1.** A set function \(f(E)\) is called super-additive if
(a) when \(f(A), f(A_\nu)\) are defined and \(A_\nu \subset A\) for all \(\nu\), and the sets \(A_\nu\) are mutually disjoint,

\[
f(A) \geq \sum \nu f(A_\nu)
\]

and
(b) \(f(\emptyset)\) is defined and equals zero (\(\emptyset\) is the empty set).

**Definition 2.** If \(A_1, A_2, \ldots, A_n\) are a collection of sets for which \(f(E)\) is defined whenever \(E\) is the intersection of \(k\) of \(A_\nu\), for \(k = 1, 2, \ldots, n\) (\(f(\emptyset)\) is defined as zero), the Sylvester sum \(S_f(A_1, A_2, \cdots, A_n)\) of \(f\) over the sets \(A_\nu\) is

\[
S_f(A_1, A_2, \cdots, A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq m_1 < m_2 < \cdots < m_k \leq n} f\left(\bigcap_{r=1}^{k} A_{m_r}\right).
\]

**Definition 3.** A set function, \(f(E)\), is super-additive in the sense of Sylvester (super-additive-S) if when \(A_\nu \subset A\) (\(\nu = 1, 2, \ldots, n\)) and \(f(A)\) and the Sylvester sum \(S_f(A_1, A_2, \cdots, A_n)\) are defined, then

\[
f(A) \geq S_f(A_1, A_2, \cdots, A_n).
\]

Clearly, functions which are super-additive-S are super-additive. The converse is not true. As an example consider three mutually disjoint sets \(A_1, A_2, A_3\) and let \(f\) be defined by

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\(^1\) Numbers in brackets refer to the bibliography at the end of the paper.

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\[ f(A_v) = 1, \quad \nu = 1, 2, 3, \]
\[ f(A_v \cup A_\mu) = 4, \quad \nu, \mu = 1, 2, 3, \]
\[ f(A_1 \cup A_2 \cup A_3) = 6, \]
\[ f(\phi) = 0. \]

**Definition 4.** By a regular domain we mean a bounded, open, simply-connected plane set whose boundary consists of a finite number of analytic arcs.

**Definition 5.** By an *admissible set* we mean a bounded open set whose components are all regular domains. The empty set is admissible.

**Definition 6.** By the stress function of an admissible set \( E \), we mean the (unique) function \( u(p; E) \), which satisfies

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \Delta u = -2, \quad p \in E, \\
u &= 0, \quad p \notin E.
\end{align*}
\]

By \( p \) we mean a point \((x, y)\).

**2. Theorem.** Let \( E_1, E_2 \) be admissible sets such that \( E_1 \subseteq E \) \((\nu = 1, 2, \ldots, n)\). Then

\[ u(p; E) = u(E) \geq \sum_{\nu=1}^{n} E_\nu. \]

**Proof.** The proof proceeds by induction on \( n \). Let us assume that Assertion (1) is true for \( n = K - 1 \), and consider the case \( n = K \) \((K > 1)\). Let \( E_1, E_2, \ldots, E_K \) be admissible sets all contained in an admissible set \( E \). We define the order of a point \( p \) as the maximal number of the sets \( E_\nu \) which contain \( p \). Clearly, the order is a non-negative integer \( \leq K \).

Consider first those points of the plane whose order is less than \( K \). Let \( P_0 \) be such a point. Then there exists one set \( E_\nu \) for which \( p_0 \in E_\nu \). Hence, \( u(p_0; E_\nu) = 0 \) and if \( E' \) is an admissible set such that \( E' \subseteq E \), \( u(p_0; E') = 0 \). If \( \{ E_1, \ldots, E_\nu^*, \ldots, E_K \} \) denote the collection of \( K - 1 \) sets not including \( E_\nu \) we have \( \sum_{i=1}^{\nu-1} E_i, \ldots, E_\nu^*, \ldots, E_K \) at \( p = p_0 \). But by our inductive hypothesis,

\[ u(p_0; E) \geq \sum_{\nu=1}^{\nu-1} E_i, \ldots, E_\nu^*, \ldots, E_K \]

Hence, Assertion (1) is true for points of order \( < K \).

It remains to consider points of order \( K \). Let \( \Gamma \) be the set of points
whose order is $K$. Since $\Gamma$ is the intersection of $K$ admissible sets, it is admissible and hence open.

If $\Gamma$ is not empty, its boundary $B(\Gamma)$ consists of points of order $< K$. Let $p_0$ be a point of $\Gamma$ and let the component of $\Gamma$ containing $p_0$ be denoted by $\Gamma_1$. In $\Gamma_1$, we have

$$\Delta \{u(E) - S_u(E_1, \cdots, E_K)\} = -2 \left\{1 - \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j}\right\} = 0.$$ 

But on $B(\Gamma_1)$, $u(E) - S_u(E_1, E_2, \cdots, E_K) \geq 0$. Hence

$$u(E) - S_u(E_1, E_2, \cdots, E_K) \geq 0$$

in $\Gamma_1$, and hence in $\Gamma$.

To complete the induction consider the case $n=1$. In this case $S_u(E_1) = u(E_1)$ and we must show that $u(E) \geq u(E_1)$. But

$$u(E) - u(E_1)$$

is clearly non-negative for $p \in E_1$, and is harmonic in $E_1$, and hence non-negative. This completes the proof.

3. Remarks. The theorem in §2 can be extended in several ways. The admissible sets can be defined in Euclidean space of dimension greater than 2. Another extension is to consider the stress function $u(\rho; D, g)$ of a domain $D$ with respect to a function $g(\rho)$ defined by

(a) $\Delta u = g, \quad \rho \in D,$

(b) $u = 0, \quad \rho \notin D.$

This function $g(\rho)$ is to be independent of $D$ and the theorem will hold if $g(\rho)$ is negative and of class $C^{(1)}$. This paper represents a portion of the author's thesis (Stanford, 1951). Some of the research was performed with the help of the ONR.

**Bibliography**


**The Johns Hopkins University**