REAL-VALUED FUNCTIONS ON PARTIALLY ORDERED SETS
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It is known that if $P$ is a partially ordered set, then $P$ can be imbedded into an everywhere branching partially ordered set $Q$ in such a manner that if a function has a limit $L$ on $P$, the function can be extended to $Q$ and have a limit $L$ on $Q$.\(^1\) The purpose of this note is to show that $P$ can be imbedded isomorphically into an everywhere branching partially ordered set $Q$ and each function $f$ on $P$ extended to $Q$, in such a manner that $f$ has a limit $L$ on $P$ if and only if it has a limit $L$ on $Q$.\(^2\)

By a partially ordered set is meant a set of elements $P = \{p\}$, with a binary relation "≤" which has the three properties:

1. $p ≤ p$ for each element $p$ of $P$;
2. if $p_1 ≤ p_2$ and $p_2 ≤ p_3$, then $p_1 ≤ p_3$; and
3. if $p_1 ≤ p_2$ and $p_2 ≤ p_1$, then $p_1 = p_2$ (identity).

As usual, "$p_1 < p_2$" will mean that $p_1 ≤ p_2$, but $p_1$ is not identical with $p_2$. An element $p_0$ of $P$ is called a minimal (maximal) element of $P$ if there is no element $p$ of $P$ for which $p < p_0$ ($p > p_0$). The only partially ordered sets which are considered hereafter are those which have no minimal elements. A partially ordered set is directed if, for each pair of elements in $P$, $p_1$ and $p_2$, an element $p_3$ can be found for which $p_1 ≤ p_3$, $i = 1, 2$. A partially ordered set is everywhere branching if, to each element $p_i$ of $P$, there corresponds a pair of elements $p_2$ and $p_3$ such that $p_i ≤ p_2$, $i = 1, 2, 3$, and

$$\{p \mid p ≤ p_2\} \cap \{p \mid p ≤ p_3\} = \emptyset$$

where $\emptyset$ is the empty set. A subset $Q = \{q\}$ of $P$ is a coinitial subset of $P$ if to each element $p$ in $P$, there corresponds an element $q$ in $Q$ such that $q ≤ p$. $Q = \{q\}$ is a residual subset of $P$ if, for each element $q$ in $Q$, $\{p \mid p ≤ q, p ∈ P\}$ is a subset of $Q$.

A single, real-valued function $f$ defined on a partially ordered set $P = \{p\}$ has a limit $L$ if, to each element $p_0$ of $P$, and $\epsilon > 0$, there corresponds an element $p_1(p_0, \epsilon)$ of $P$ such that $p_1 ≤ p_0$ and $|f(p) - L| < \epsilon$ for $p ≤ p_1$.\(^3\)

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\(^2\) The author wishes to thank the referee for his general suggestions, particularly in the simplification of the proof of Theorem 1.

\(^3\) See Alaoglu and, Birkhoff, General ergodic theorems, Ann. of Math. vol. 41 (1940) pp. 293–309.
A single, real-valued function \( f \) defined on a partially ordered set \( P \) has a partial limit \( L \) on \( P \) if, for some residual subset \( Q \) of \( P \), the function \( f \), considered as a function on \( Q \), has a limit \( L \).

**Theorem 1.** A partially ordered set \( P = \{ p \} \) may be imbedded into an everywhere branching partially ordered set \( Q = \{ q \} \) by an isomorphism \( g \). Furthermore, a function \( h \) of \( Q \) onto \( g(P) \) can be found which has the following two properties:

1. \( h[g(p)] = g(p) \) for each \( p \) in \( P \); and
2. \( f \) being any real function on \( P \), then the function \( f_* \), which is defined by (a) \( f_*[g(p)] = f(p) \) for \( p \) in \( P \), and (b) \( f_*[g(q)] = f_*[h(q)] \) for \( q \) in \( Q \), has a limit \( L \) on \( Q \) if and only if \( f \) has a limit \( L \) on \( P \).

**Proof.** For each \( p \) in \( P \) let \( g(p) = \{ x \mid x \leq p, x \in P \} \). Let

\[ Q = \{ q \mid q \text{ is a coinitial subset of } g(p), p \in P \}, \]

and order the elements of \( Q \) by set inclusion. To see that \( Q \) is an everywhere branching partially ordered set, well order the elements of \( P \) into a sequence, say \( \{ r_\xi \}, \xi < \gamma \). A simply ordered subset \( A = \{ a \} \) of a partially ordered set \( B = \{ b \} \) shall be called a path (in \( B \)) if there is no element \( b_0 \) of \( B \) such that \( b_0 \leq a \) for each element \( a \) in \( A \). Clearly, if \( b_0 \) is any element of \( B \), then there exists a path in \( B \),

\[ a_0 > a_1 > \cdots > a_\xi > \cdots \]

Now let \( q = \{ y \} \) be any element of \( Q \). Let \( y_0^0 \) be the first element of \( q \) and

\[ y_0^0 > y_1^0 > \cdots > y_\xi^0 > \cdots \]

be any path \( Z_0 \) in \( q \) for which \( y_\xi^0 \) is a maximal element. This is possible since \( q \) is a partially ordered set with no minimal element. Denote by \( A_0 \) the set

\[ A_0 = \{ y \mid y \in q, y \preceq y_0^0, y_\xi^0 \in Z_0 \}. \]

We continue by transfinite induction. Suppose that the paths \( Z_\mu \) and the sets \( A_\mu \) have been defined for \( \mu < \lambda \). Let \( y_0^\lambda \) be the first element of \( q - \bigcup_{\mu < \lambda} A_\mu \). Let

\[ y_0^\lambda > y_1^\lambda > \cdots > y_\xi^\lambda > \cdots \]

be any path \( Z_\lambda \) in \( q \) for which \( y_\xi^\lambda \) is a maximal element. This is possible since \( q - \bigcup_{\mu < \lambda} A_\mu \) is a partially ordered set with no minimal element.

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4 A mapping \( g \) of a partially ordered set \( (P, \leq^1) \) into a partially ordered set \( (Q, \leq^2) \) is an isomorphism if \( g \) is one-to-one, and \( p_1 \leq^1 p_2 \) if and only if \( g(p_1) \leq^2 g(p_2) \).
Denote by $A_\lambda$ the set

$$A_\lambda = \{ y \mid y \in q, y \preceq y_0^\lambda, y_0^\lambda \in Z_\lambda \}.$$ 

Consider the two subsets of $q$

$$q_1 = \{ y_{\alpha+2n} \mid \alpha = 0 \text{ or a limit number, } \lambda \text{ any ordinal} \}$$

and

$$q_2 = \{ y_{\alpha+3n} \mid \alpha = 0 \text{ or a limit number, } \lambda \text{ any ordinal} \}.$$ 

Each of the two sets are coinitial in $q$. Therefore $q_1$ and $q_2$ are elements of $Q$.

Let $g$ be the function which takes each point $p$ of $P$ into the element $g(p)$ of $Q$. Clearly $g$ is an isomorphism of $P$ into $Q$. Let $h$ be the function which is defined by: (1) $h(q) = g(p_0)$ if $p_0$ is the unique maximal element of $q$; and (2) $h(q) = g(p_0)$, where $p_0$ is the first element of $q$, if $q$ has no unique maximal element. If $f$ is a real function on $P$, then denote by $f_*$ the function which is defined by $f_*(g(p)) = f(p)$ for $p$ in $P$, and $f_*(q) = f_*(h(q))$.

Suppose that $f_*$ has a limit $L$ on $Q$. For any element $p_0$ of $P$, and $\epsilon > 0$, denote by $E_*(p_0)$ the set

$$E_*(p_0) = \{ p \mid p \preceq p_0 \text{ and } |f(p) - L| \geq \epsilon \}.$$ 

If $E_*(p_0)$ were to be a coinitial subset of $g(p_0)$, then the relation $|f_*(g(q)) - L| \geq \epsilon$ would be true for all $q \leq E_*(p_0)$. But this would contradict the function $f_*$ having a limit $L$ on $Q$. Therefore $E_*(p_0)$ is not a coinitial subset of $g(p_0)$. Consequently, for some element $p_1$ of $g(p_0)$, i.e., $p_1 \preceq p_0$, we have $\{ p \mid p \preceq p_1 \} \cap E_*(p_0) = \emptyset$. Hence $|f(p) - L| < \epsilon$ for $p \preceq p_1$. Thus $f$ has a limit $L$ on $P$.

Now suppose that the function $f$ has a limit $L$ on $P$. Let $q_0$ be any element of $Q$, and $\epsilon > 0$. Let $p_0$ be any element of $q_0$. For some element $p_1$ of $P$, where $p_1 \preceq p_0$, we have $|f(p) - L| < \epsilon$ for $p \preceq p_1$. If $q_1 = \{ p \mid p \preceq p_1, p \in q_0 \}$, then $q_1$ is a coinitial subset of $g(p_1)$. Thus $q_1$ is an element of $Q$. If $q \preceq q_1$, then $|f_*(g(q)) - L| < \epsilon$. Therefore the function $f_*$ has a limit $L$ on $Q$.

When the partially ordered set $P$ is directed, one can study the behavior on a coinitial subset of $P$ of a real function $f$ defined on $P$, by inspecting the everywhere branching partially ordered set $Q$ and the function $f_*$ which are obtained from the previous theorem. Specifically we have

**Theorem 2.** Let $P$ be a directed partially ordered set and $f$ a real function defined on it. Let $Q$ and $f_*$ be the same as in Theorem 1.
Then a necessary and sufficient condition that $f$ have a limit $L$ on some coinitial subset $q$ of $P$ is that $f_*$ have a partial limit $L$ on $Q$.

**Proof.** The necessity is trivial. Consider the sufficiency. Let $Z$ be a residual subset of $Q$ on which $f_*$ has a limit $L$. Let $q_0$ be an element of $Z$. $q_0$ is a coinitial subset of $P$. Let $\epsilon > 0$ and $p_0$ be any element of $q_0$. Let

$$q_1 = \{ p \mid p \leq p_0, p \in q_0 \}.$$ 

Suppose that for each point $p_1$ of $q_1$, a point $p_2$ of $q_1$, where $p_2 \leq p_1$, can be found so that $|f(p_2) - L| \geq \epsilon$. Since $P$ is directed, and thus also $q_0$ and $q_1$,

$$q_2 = \{ p \mid p \in q_1, |f(p) - L| \geq \epsilon \}$$

is a coinitial subset of $P$. Furthermore, $|f_*(q) - L| \geq \epsilon$ for $q \leq q_1$. Thus $f_*$ cannot have the limit $L$ on $Z$. From this contradiction we see that for some $p_1 \leq p_0$, $|f(p) - L| < \epsilon$ for $p \leq p_1$. Consequently $f$ has the limit $L$ on $q_0$.

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