A UNIFORM APPROXIMATION TO FOURIER COEFFICIENTS

M. L. JUNCOSA AND D. M. YOUNG

Let \( f(x) \) be a real-valued function which is continuous in the interval \( I: -1 \leq x \leq 1 \) except at \( N \) points at which it may have a finite jump. The sequence of its ordinary complex Fourier coefficients is given by

\[
a_n = \frac{1}{2} \int_{-1}^{1} f(x)e^{-inx}dx, \quad n = 0, \pm 1, \ldots.
\]

It is the purpose of this note to show that the elements of the sequence of Riemann sum approximations to \( a_n \),

\[
b_n(M) = \left\{ \begin{array}{ll}
\frac{1}{2M} \sum_{j=-M+1}^{M} f\left(\frac{j}{M}\right)e^{-inx/M}, & n = 0, \pm 1, \ldots, \pm M, \\
0, & |n| > M
\end{array} \right.
\]

tend to \( a_n \) uniformly as \( M \to \infty \).

Let

\[
s_m = \sum_{j=-m}^{m} a_j e^{inx}, \quad m = 0, 1, \ldots,
\]

and

\[
\sigma_k(x) = \frac{1}{k+1} \sum_{m=0}^{k} s_m(x), \quad k = 0, 1, \ldots.
\]

Then

\[
\sigma_k(x) = \sum_{n=-k}^{k} A_{k,n} e^{inx}
\]

where

\[
A_{k,n} = \begin{cases} 
(1 - |n|/k + 1)a_n, & n = 0, \pm 1, \ldots, \pm k, \\
0, & |n| > k.
\end{cases}
\]

Let us cover the set of points of discontinuity of \( f(x) \) in \( I \) by a set \( E \) which is the sum of a finite number of open intervals of lengths \( \eta_i, i = 1, 2, \ldots, N \), each of which covers at most one point of discontinuity and such that \( \sum_{i=1}^{N} \eta_i = \eta < \varepsilon/6F \), where \( F = \text{LUB}|f(x)| \) in 

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373
I. Then, on \( I - IE \), by a generalization of Fejér’s theorem on summability \((C, 1)\) of Fourier series (see, e.g., [1, p. 66]) \( \sigma_k(x) \) converges uniformly to \( f(x) \). Therefore, let \( M_1 \) be such that \( | \sigma_k(x) - f(x) | < \varepsilon/6 \) uniformly on \( I - IE \) for \( k \geq M_1 \). Henceforth, let \( M \geq \max \{ M_1, 3NF/\varepsilon \} \). Since, in general, \( | \sigma_k(x) | \leq F \) (see [1, p. 58]), on \( IE \) we have \( | \sigma_k(x) - f(x) | \leq 2F \).

For \(|n| \leq M\), using the above inequalities, we obtain

\[
|b_n(M) - A_{M,n}| = \frac{1}{2M} \left| \sum_{j=-M}^{M} \left[ f \left( \frac{j}{M} \right) - \sigma_M \left( \frac{j}{M} \right) \right] e^{-in\pi j/M} \right|
\leq \frac{1}{2M} \left| \sum_{j=-M}^{M} \left[ f \left( \frac{j}{M} \right) - \sigma_M \left( \frac{j}{M} \right) \right] \right|
\leq \frac{1}{2M} \left[ \frac{2M\varepsilon}{6} + \sum_{i=1}^{N} (M\eta_i + 1)2F \right] < \frac{2\varepsilon}{3}.
\]

For \(|n| > M\), \( |b_n(M) - A_{M,n}| = 0 \). For all \( n \), we have

\[
|A_{M,n} - a_n| = \frac{1}{2} \left| \int_{-1}^{1} \left[ \sigma_M(x) - f(x) \right] e^{-inx} dx \right|
< \frac{1}{2} \left( \frac{\varepsilon}{3} + 2\eta \right) < \frac{\varepsilon}{3}.
\]

Putting these results in the triangle inequality, we get \( |b_n(M) - a_n| < \varepsilon \) uniformly for all integral \( n \).

**Reference**


Ballistic Research Laboratories, Aberdeen Proving Ground, and University of Maryland