ON THE SEPARATION OF SPECTRA

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Let a positive $p(t)$ and a real-valued $q(t)$, where $-\infty < t < \infty$, be continuous functions with the property that the differential equation

$$(1) \qquad (px')' + (q + \lambda)x = 0$$

is of the "limit point" type for both $t = -\infty$ and $t = \infty$. Then (1) determines a boundary value problem in the Hilbert space $L^2(-\infty, \infty)$; that is, $L(z) = -(pz')' - qz$ is a self-adjoint operator defined on the set of functions $z(t)$ for which z, pz' are absolutely continuous; and z, $L(z)$ are of class $L^2(-\infty, \infty)$. Let the spectrum of this operator be denoted by S and its derived set by S'. The assumptions above also imply that (1) and a boundary condition

$$(2) \qquad x(0) = 0$$

determine two boundary value problems, one in each of the Hilbert spaces $L^2(-\infty, 0)$ and $L^2(0, \infty)$. Let S_1 and S_2 denote the spectra of the respective problems. Then the following theorem will be proved:

Let λ' and λ'' be a pair of points contained in one of the sets S, $S_1 + S_2$. Then the other set contains at least one point in the closed interval $[\lambda', \lambda'']$.

PROOF. Since it has been shown [1, p. 714] that $S' = S_1' + S_2'$, it is necessary to consider only the cases when λ', λ'' are isolated points of the point spectrum. It may also be assumed that $\lambda' = -\lambda$, $\lambda'' = \lambda$ for the addition of the constant $(\lambda' + \lambda'')/2$ to $q(t)$ simultaneously translates the sets S, S_1, S_2 by $-(\lambda' + \lambda'')/2$.

Assume firstly that $-\lambda$, λ are isolated points of S_2 and let x_1, x_2 be the corresponding eigenfunctions. Put

$$(3) \qquad z(t) = c_1x_1 + c_2x_2 \quad \text{or} \quad z(t) = 0$$

according as $0 \leqq t < \infty$ or $-\infty < t < 0$, where c_1, c_2 are chosen so that $z'(-0) = 0$. Also choose the normalization $\int_{-\infty}^{\infty} z^2 dt = 1$. Then z is in the domain of the operator $L(z)$. Thus

$$(4) \qquad \int_{-\infty}^{\infty} L^2(z)dt = \lambda^2 \int_0^{\infty} (c_1x_1 - c_2x_2)^2 dt = \lambda^2 \int_{-\infty}^{\infty} z^2 dt = \lambda^2$$

in view of the orthogonality of x_1, x_2 on $(0, \infty)$. But if $[-\lambda, \lambda]$ is

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free of points of S the spectral resolution of the operator $L(z)$ implies

$$\int_{-\infty}^{\infty} L^2(z)dt > \lambda^2 \int_{-\infty}^{\infty} z^2 dt = \lambda^2 \tag{5}$$

which contradicts (4) and shows that S contains a point in $[-\lambda, \lambda]$.

Clearly the same proof applies if $-\lambda, \lambda$ are points of S_1. Now let $-\lambda \in S_1$ and $\lambda \in S_2$ where y_1, y_2 are corresponding eigenfunctions. In this case put

$$z(t) = c_1 y_1 \quad \text{or} \quad z(t) = c_2 y_2 \tag{6}$$

according as $-\infty < t \leqq 0$ or $0 \leqq t < \infty$, where c_1, c_2 are chosen so that $c_1 y_1'(0) = c_2 y_2'(0)$. Then $z(t)$ is in the domain of the operator $L(z)$, and

$$\int_{-\infty}^{\infty} L^2(z)dt = \lambda^2 \int_{-\infty}^{\infty} z^2 dt \tag{7}$$

which again implies that the interval $[-\lambda, \lambda]$ contains a point of S.

Finally if $-\lambda, \lambda$ are points of S and w_1, w_2 are corresponding eigenfunctions, put

$$z(t) = c_1 w_1 + c_2 w_2 \tag{8}$$

for $-\infty < t < \infty$, where c_1, c_2 are chosen so that $z(0) = 0$. Then z is in the domain of the operators which determine the sets S_1, S_2. Now, if we assume that the interval $[-\lambda, \lambda]$ contains no points of either S_1 or S_2, the spectral theorem applied to the operators determining these two sets yields a contradiction as before. This completes the proof.

Reference

1. K. G. Wolfson *On the spectrum of a boundary value problem with two singular endpoints*, Amer. J. Math. vol. 72 (1950) pp. 713–719.

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