ON THE SEPARATION OF SPECTRA

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Let a positive \( p(t) \) and a real-valued \( q(t) \), where \(-\infty < t < \infty\), be continuous functions with the property that the differential equation

\[
(p x')' + (q + \lambda)x = 0
\]

is of the "limit point" type for both \( t = -\infty \) and \( t = \infty \). Then (1) determines a boundary value problem in the Hilbert space \( L^2(-\infty, \infty) \); that is, \( L(z) = -(pz')' - qz \) is a self-adjoint operator defined on the set of functions \( z(t) \) for which \( z, pz' \) are absolutely continuous; and \( z, L(z) \) are of class \( L^2(-\infty, \infty) \). Let the spectrum of this operator be denoted by \( S \) and its derived set by \( S' \). The assumptions above also imply that (1) and a boundary condition

\[
x(0) = 0
\]

determine two boundary value problems, one in each of the Hilbert spaces \( L^2(-\infty, 0) \) and \( L^2(0, \infty) \). Let \( S_1 \) and \( S_2 \) denote the spectra of the respective problems. Then the following theorem will be proved:

Let \( \lambda' \) and \( \lambda'' \) be a pair of points contained in one of the sets \( S, S_1 + S_2 \). Then the other set contains at least one point in the closed interval \( [\lambda', \lambda''] \).

Proof. Since it has been shown \([1, p. 714]\) that \( S' = S'_1 + S'_2 \), it is necessary to consider only the cases when \( \lambda', \lambda'' \) are isolated points of the point spectrum. It may also be assumed that \( \lambda' = -\lambda, \lambda'' = \lambda \) for the addition of the constant \( (\lambda' + \lambda'')/2 \) to \( q(t) \) simultaneously translates the sets \( S, S_1, S_2 \) by \( -(\lambda' + \lambda'')/2 \).

Assume firstly that \(-\lambda, \lambda \) are isolated points of \( S_2 \) and let \( x_1, x_2 \) be the corresponding eigenfunctions. Put

\[
z(t) = c_1 x_1 + c_2 x_2 \quad \text{or} \quad z(t) = 0
\]

according as \( 0 \leq t < \infty \) or \(-\infty < t < 0\), where \( c_1, c_2 \) are chosen so that \( z'(-0) = 0 \). Also choose the normalization \( \int_{-\infty}^{\infty} z^2 \, dt = 1 \). Then \( z \) is in the domain of the operator \( L(z) \). Thus

\[
\int_{-\infty}^{\infty} L^2(z) \, dt = \lambda^2 \int_{0}^{\infty} (c_1 x_1 - c_2 x_2)^2 \, dt = \lambda^2 \int_{-\infty}^{\infty} z^2 \, dt = \lambda^2
\]
in view of the orthogonality of \( x_1, x_2 \) on \( (0, \infty) \). But if \([-\lambda, \lambda]\) is

Presented to the Society, April 25, 1953; received by the editors October 20, 1952.

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free of points of $S$ the spectral resolution of the operator $L(z)$ implies

$$\int_{-\infty}^{\infty} L^2(z)dt > \lambda^2 \int_{-\infty}^{\infty} z^2 dt = \lambda^2$$

which contradicts (4) and shows that $S$ contains a point in $[-\lambda, \lambda]$.

Clearly the same proof applies if $-\lambda, \lambda$ are points of $S_1$. Now let $-\lambda \in S_1$ and $\lambda \in S_2$ where $y_1, y_2$ are corresponding eigenfunctions. In this case put

$$z(t) = c_1 y_1 \quad \text{or} \quad z(t) = c_2 y_2$$

according as $-\infty < t \leq 0$ or $0 \leq t < \infty$, where $c_1, c_2$ are chosen so that $c_1 y_1'(0) = c_2 y_2'(0)$. Then $z(t)$ is in the domain of the operator $L(z)$, and

$$\int_{-\infty}^{\infty} L^2(z)dt = \lambda^2 \int_{-\infty}^{\infty} z^2 dt$$

which again implies that the interval $[-\lambda, \lambda]$ contains a point of $S$.

Finally if $-\lambda, \lambda$ are points of $S$ and $w_1, w_2$ are corresponding eigenfunctions, put

$$z(t) = c_1 w_1 + c_2 w_2$$

for $-\infty < t < \infty$, where $c_1, c_2$ are chosen so that $z(0) = 0$. Then $z$ is in the domain of the operators which determine the sets $S_1, S_2$. Now, if we assume that the interval $[-\lambda, \lambda]$ contains no points of either $S_1$ or $S_2$, the spectral theorem applied to the operators determining these two sets yields a contradiction as before. This completes the proof.

Reference


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