THE FUNDAMENTAL GROUP OF THE PRINCIPAL
COMPONENT OF A COMMUTATIVE
BANACH ALGEBRA

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We consider an arbitrary commutative Banach algebra over the
complex numbers. Let $B$ denote the algebra, $\{a, b, c, x, y, z, \ldots\}$
its elements, and $\{\lambda, \mu, \nu, \ldots\}$ complex numbers. We assume that $B$
contains a unit element, $e$, with $\|e\| = 1$.

If $a^{-1}$ exists ($aa^{-1} = a^{-1}a = e$), the element $a$ is called "regular." The
set of regular elements will be denoted by $G$. It is well known that
(1) $G$ is a topological group relative to multiplication and (2) $G$ is
an open subset of $B$. Since $G$ is open, it is a union of maximal open
connected sets, its components. We call the component $G_1$ containing
the unit $e$ the "principal component" [1]. It is easy to see that $G_1$ is a
subgroup of $G$.

The function $\exp (x) = e + \sum_1^\infty \frac{x^n}{n!}$ is defined for all $x$ in $B$ and
has the usual properties of the classical exponential function. If we
let $\pi_1(G_1)$ denote the fundamental group of $G_1$, we may state our
main result as follows:

**Theorem 1.** Let $P = \{x | \exp (x) = e\}$. $P$ is an additive group which is
isomorphic to $\pi_1(G_1)$.

We shall give a complete proof based on Schreier’s theory of the
universal covering group and then we shall outline a second proof
which depends only on results from the theory of Banach algebras.\(^2\)
The result of Schreier which we shall use may be stated as follows [2]:

**Theorem.** Let $B$ be a simply-connected, locally-connected and locally
simply-connected topological group. If $P$ is a discrete normal subgroup
of $B$, then the fundamental group of the topological space $B/P$ is iso-
morphic to the group $P$.

The algebra $B$, regarded as an additive group with the metric
topology of the norm, clearly satisfies the hypotheses of Schreier’s

\(^1\) This result is contained in the author's doctoral dissertation, *The theory of
analytic functions in Banach algebras*, completed in June 1952 under E. R. Lorch at
Columbia University.

\(^2\) The second proof is the one used in the author’s dissertation. It was later pointed
out by S. Eilenberg that a different proof is possible if one brings the Schreier theory
to bear.
Theorem. The set \( P = \{ x \mid \exp(x) = e \} \) is obviously a normal subgroup of \( B \) since \( \exp(x-y) = \exp(x) \exp(-y) = \exp(x) [\exp(y)]^{-1} \). Further, \( \exp(x) \) maps \( B \) onto the principal component, \( G_1 \). It is clearly a continuous map. It is also an open mapping since its inverse, \( \log y \), is continuous. Thus, \( \exp(x) \) is an open homomorphism of the additive topological group \( B \) onto the multiplicative topological group \( G_1 \). The kernel of this homomorphism is \( P \), so that \( B/P \) is isomorphic to \( G_1 \) as topological groups. Hence \( \pi_1(B/P) \) is isomorphic to \( \pi_1(G_1) \). If it can be shown that \( P \) is discrete, the Schreier theorem is applicable and Theorem 1 follows immediately.

To prove \( P \) discrete it suffices to show that \( 0 \) is an isolated element of \( P \). Suppose there is a sequence of elements \( \{ z_n \} \) such that \( z_n \in P \), \( z_n \neq 0 \) and \( \lim z_n = 0 \). By a theorem of Lorch [1], this implies that \( z_n = 2\pi i \sum_{j=1}^{n} n e_j \), where the \( e_j \) are idempotent elements \( (e_j^2 = e_j) \) and the \( n_j \) are rational integers. Furthermore, the spectrum of \( z_n \) consists of the points \( 2\pi i n_1, \ldots, 2\pi i n_k \). However, \( \lim ||z_n|| = 0 \) and, as is well known, the spectrum of \( z_n \) contains no points exterior to the circle of radius \( ||z_n|| \), center at the origin. For sufficiently large \( n \), this means that \( n_j = 0 \) for \( j = 1, \ldots, k \); i.e. \( z_n = 0 \). This contradiction completes the proof.

Now we shall indicate a more elementary and constructive proof in which no recourse is had to the Schreier theory.

First we establish a lemma concerning the function \( \exp(x) \).

**Lemma:** Let \( w_0 = \exp(x_0) \) and \( 0 < e < 1/||w_0^{-1}|| \). Let

\[
\delta = \sum_{1}^{\infty} \left( ||w_0^{-1}||e^n \right).
\]

If \( E \) is the set \( \{ w \mid ||w-w_0|| < e \} \), then for every \( w \) in \( E \) there is an \( x \) such that \( ||x-x_0|| < \delta \) and \( \exp(x) = w \).

**Proof.** Choosing \( w = w_0 + b \) where \( ||b|| < e \), we have \( ||w_0^{-1}w-e|| = ||w_0^{-1}b|| \leq ||w_0^{-1}|| ||b|| < 1 \). Hence \( w_0^{-1}w \) is in \( G_1 \) and there is an element \( c \) in \( B \) such that \( \exp(c) = w_0^{-1}w \). In fact, we may take

\[
c = \sum_{1}^{\infty} \left( (1/e) (e - w_0^{-1}w)^n \right)
\]

so that \( ||c|| < \sum_{1}^{\infty} (1/e) (||w_0^{-1}||)^n = \delta \). The element \( x = x_0 + c \) satisfies the conclusion of the lemma. It is important to note that \( \delta \) approaches \( 0 \) as \( e \) approaches 0.

Using this lemma, we are able to prove

**Theorem 2.** Let \( K: \{ f(s), 0 \leq s \leq 1 \} \) be a curve in \( G_1 \) joining \( e = f(0) \)
to \( w = f(1) \). There exists an element \( z \) in \( B \) such that \( \exp(z) = w \) and the curve \( K(z) : \{ \exp(ts), 0 \leq t \leq 1 \} \) is homotopic to \( K \) in \( G_1 \).

**Proof.** If \( z_s \) is such that \( \exp(z_s) = f(s) \), let \( K(z_s) \) denote the curve \( \{ \exp(ts), 0 \leq t \leq 1 \} \). \( K(z_s) \) is a compact set in \( G_1 \). Hence, there is a number \( \rho > 0 \) such that every sphere with center on \( K(z_s) \) and radius \( \rho \) is contained in \( G_1 \). Choose \( \varepsilon \) such that \( 0 < \varepsilon < \min \{ \rho, 1 \} \). There is a number \( \gamma(\varepsilon) > 0 \) such that \( ||f(r) - f(s)|| < \varepsilon \) whenever \( |r - s| < \gamma(\varepsilon) \).

By the lemma, there is an element \( z_r \) such that \( ||z_r - z_s|| < \delta \) and \( \exp(z_r) = f(r) \), where \( \delta = \sum_1^n (1/n)(||f(s)||^{-1}||e||)^n \), that is, \( z_r = z_s + b \) where \( ||b|| < \delta \).

For all \( t, 0 \leq t \leq 1 \), \( ||\exp(te) - \exp(te_s)|| \leq ||\exp(te)|| \cdot ||\exp(b) - e|| \leq \sum_0^n (1/n!)||z_s||^n \cdot \sum_1^n (1/n!)||b||^n \leq \exp ||z_s|| \cdot \sum_1^n \delta^n/n! \). By choosing \( \varepsilon \) sufficiently small, thereby making \( \delta \) small, we have \( ||\exp(te) - \exp(te_s)|| < \rho \). It follows that \( K(z_s) \) is homotopic in \( G_1 \) to the curve \( K(z_s) \cup K_b^s \) consisting of \( K(z_s) \) followed by the arc \( K_b^s \) of \( f \) from \( s = u \leq u \leq 0 \).

In particular, since \( \exp(0) = f(0) = e \), there is an \( r > 0 \) and an element \( z_r \) such that \( K(z_r) \) is homotopic to the arc \( K_b^0 \) of \( f \) from \( 0 \leq u \leq s \leq r \).

The set of all real numbers \( r \) for which this holds has a least upper bound, \( \mu \leq 1 \). Suppose \( \mu < 1 \). We obtain a contradiction.

Let \( \exp(z_s') = f(\mu) \). There is an element \( z_s' \) such that \( K(z_s') \) is homotopic to \( K(z_s') \cup K_b^s \), \( s < \mu \). But there is a \( z_s \) such that \( K(z_s) \) is homotopic to \( K_b^0 \). Since \( \exp(z_s') = \exp(z_s) = f(s) \), we have \( z_s = z_s' + c \) where \( \exp(c) = e \). Let \( z_s = z_s' + c \). It is simple to show that \( K(z_s) \) is homotopic to \( K(z_s) \cup K_\ast \), which is, in turn, homotopic to \( K_b^s \cup K_b^s = K_b^s \). Hence \( K(z_s) \) is homotopic to \( K_b^s \). By the same reasoning, there is a number \( s_\ast > \mu \) such that \( K_b^s \) is homotopic to \( K(z_s) \), contradicting the assumption on \( \mu \). Therefore, \( \mu = 1 \).

The next theorem then follows easily.

**Theorem 3.** Every closed curve in \( G_1 \) with \( e \) as initial and end point is homotopic in \( G_1 \) to a curve \( K(b) \) of the form \( \{ \exp(tb), 0 \leq t \leq 1, \exp(b) = e \} \). If \( \exp(b') = e \) and \( b \neq b' \), then \( K(b') \) is not homotopic to \( K(b) \). Thus each homotopy class contains precisely one curve of the form \( \{ \exp(tb), 0 \leq t \leq 1 \} \) where \( \exp(b) = e \).

**Proof.** The first part of the theorem is obtained by choosing any point \( w = f(s) \) on \( K \). By Theorem 2, there is a \( z \) such that \( K(z) \) is homotopic to \( K_\ast \) and a \( z' \) such that \( K(z') \) is homotopic to \( K_\ast^1 \). The element \( b = z' - z \) gives the desired result.

Noting that \( \int_{K(b')} x^{-1}dx = b \) and \( \int_{K(b')} x^{-1}dx = b' \), we see that \( K(b) \) cannot be homotopic to \( K(b') \), for in that event, the integrals would
be equal by the Cauchy integral theorem (by [1] and a result in the author's dissertation not yet published).

Theorem 1 follows directly from Theorem 3.

BIBLIOGRAPHY


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REMARK ON A FORMULA FOR THE BERNOULLI NUMBERS

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Some years ago Garabedian [1] proved the following formula:

\[ B_{k+1} = \frac{(-1)^{k+1} (k + 1)}{2^{k+1} - 1} \sum_{r=0}^{k} (-1)^{r} \frac{\Delta^{r} 2^{r+1}}{2r+1} \quad (k \geq 0), \]

where the even suffix notation is employed for the Bernoulli numbers. The proof of (1) made use of the sum of a certain divergent series.

We wish to point out that (1) is not new. It can be found (in somewhat different notation) in [3, p. 224, formula (68)].

It may be of interest to give a short proof of (1). We use the formula [2, p. 28]

\[ C_k = 2^{k+1}(1 - 2^{k+1}) \frac{B_{k+1}}{k + 1}, \]

where the \( C_k \) are the coefficients in the Euler polynomial:

\[ E_k(x) = \left( x + \frac{C_k}{2} \right)^k = \sum_{s=0}^{k} \binom{k}{s} 2^{-s} C_s x^{k-s}. \]

Then in view of

\[ E_k(x + 1) + E_k(x) = 2x^k, \]

we have

\[ E_k(x) = \left( 1 + \frac{1}{2} \Delta \right)^{-1} x^k = \sum_{s=0}^{k} (-1)^s 2^{-s} \Delta^s x^k. \]

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