

A NOTE ON THE DERIVATIONS OF LIE ALGEBRAS¹

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1. Introduction. Let L be a Lie algebra and V a subalgebra of L . If W is another subalgebra of L such that $L = V + W$ and $V \cap W = \{0\}$, we say that L splits over V and that W is a complement of V in L . If V is an ideal in L , splitting of L over V is equivalent to the existence of an isomorphism θ of L/V into L such that $\phi\theta$ is the identity on L/V where ϕ is the natural homomorphism of L onto L/V .

A linear transformation T on the vector space of a Lie algebra L is called a derivation of L if $T(x \circ y) = T(x) \circ y + x \circ T(y)$ for all x, y in L . T is said to be an inner derivation of L if there exists an element y in L such that $T(x) = x \circ y$ for all x in L . Let T_1 and T_2 be two derivations of L . Then if we define their commutation $T_1 \circ T_2$ by $(T_1 \circ T_2)(x) = (T_2 T_1 - T_1 T_2)(x)$ for all x in L , it is easy to see that the derivations of L form a Lie algebra which we denote by $D(L)$. Further, it is easy to show that under this definition of commutation the inner derivations of L form an ideal in $D(L)$ which we denote by $I(L)$.

The purpose of this note is to prove the following theorem:

THEOREM. *Let L be a Lie algebra over a ground field of characteristic 0 and let R denote the radical of L . Then, if $D(R)$ splits over $I(R)$, $D(L)$ splits over $I(L)$.*

The converse of this theorem is still an open question, but in an appendix we give an example of a nilpotent Lie algebra whose derivations do not split over the inner derivations.

2. Preliminaries. Levi's well known decomposition theorem states that any Lie algebra L splits over its radical R and that any complement S of R must be a maximal semi-simple subalgebra of L . Malcev² has shown that any two complements of the radical are conjugate under an automorphism of L which has the property that it maps ideals of L which are contained in R onto themselves. Further, Harish-Chandra³ has shown that any maximal semi-simple sub-

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² A. Malcev, C. R. Acad. Sci. URSS. vol. 36 (1942) p. 42.

³ Harish-Chandra, *On the radical of a Lie algebra*, Proc. Amer. Math. Soc. vol. 1 (1950).

algebra of L is a complement of the radical. For our purposes we need the following easy extension of these results:

2.1. LEMMA. *Let R be the radical of L and let T be an ideal of L such that $T \subseteq R$ and such that L splits over T . Then if S is any semi-simple subalgebra of L , there exists a subalgebra Q of L with the properties:*

- (1) *Q is a complement of T in L .*
- (2) *$S \subseteq Q$.*

PROOF. Let Q' be any complement of T in L . Then since $L/R = (Q'+R)/R \cong Q'/Q' \cap R$, there exists a maximal semi-simple subalgebra \mathfrak{S}' of L such that $\mathfrak{S}' \subseteq Q'$. Let \mathfrak{S} be a maximal semi-simple subalgebra of L such that $S \subseteq \mathfrak{S}$. By the above results of Malcev and Harish-Chandra, there exists a conjugacy automorphism θ of L such that $\theta(\mathfrak{S}') = \mathfrak{S}$ and $\theta(T) = T$. Hence putting $Q = \theta(Q')$ gives the result.

Let $L = S + R$ as before and let $\mathfrak{A}(S)$ denote those derivations of L which map S into (0) . It is easy to see that $\mathfrak{A}(S)$ is a subalgebra of $D(L)$ and that $\mathfrak{A}(S) \cap I(L)$ is an ideal of $\mathfrak{A}(S)$. Hochschild⁴ has shown that $D(L) = \mathfrak{A}(S) + I(L)$ whence we have:

2.2. LEMMA. *If $\mathfrak{A}(S)$ splits over $\mathfrak{A}(S) \cap I(L)$, then $D(L)$ splits over $I(L)$.*

Notation. Throughout the remainder of this note, if x is any element of a Lie algebra X , x^* will denote the inner derivation of X induced by x . If Y is any subset of X , Y^* will denote the set of inner derivations of X induced by the elements of Y .

Now define a mapping ρ as follows: $\rho(D)$ is the restriction to R of D for all D in $D(L)$. Since the radical of a Lie algebra is characteristic with respect to derivations, ρ is a homomorphism of $D(L)$ into $D(R)$. Further, the restriction of ρ to $\mathfrak{A}(S)$ is an isomorphism of $\mathfrak{A}(S)$ into $D(R)$. The mapping $x \rightarrow x^*$ is clearly a homomorphism of L into $D(L)$. Thus since S is a semi-simple subalgebra of L , S^* is a semi-simple subalgebra of $D(L)$ and $\rho(S^*)$ is a semi-simple subalgebra of $D(R)$.

2.3. LEMMA. *$\rho(\mathfrak{A}(S))$ is the centralizer H , say, of $\rho(S^*)$ in $D(R)$. $[H = \{T \mid T \text{ in } D(R); T \circ \rho(S^*) = (0)\}].$*

PROOF. If \bar{T} is in $\rho(\mathfrak{A}(S))$, take T in $\mathfrak{A}(S)$ such that $\rho(T) = \bar{T}$.

$$T(r \circ s) = T(r) \circ s + r \circ T(s) = T(r) \circ s,$$

i.e. $\bar{T}(r \circ s) = \bar{T}(r) \circ s$ for all r in R , s in S . On the other hand,

⁴ G. P. Hochschild, *Semi-simple algebras and generalized derivations*, Amer. J. Math. vol. 64 (1942).

$$\begin{aligned}\bar{T}(r \circ s) &= \bar{T}(s^*(r)) = \bar{T}(\rho(s^*)(r)), \\ \bar{T}(r) \circ s &= s^*(\bar{T}(r)) = \rho(s^*)(\bar{T}(r))\end{aligned}$$

whence $\bar{T} \circ \rho(s^*) = 0$. Therefore $\rho(\mathfrak{A}(S)) \subseteq H$. Conversely, if \bar{T} is in H , define a linear mapping T such that $T(s) = 0$, $T(r) = \bar{T}(r)$ for all s in S and all r in R . Then it is easy to see that T is a derivation and that $\rho(T) = \bar{T}$. Hence $H \subseteq \rho(\mathfrak{A}(S))$ and the lemma is proved.

2.4. LEMMA. $\rho(\mathfrak{A}(S) \cap I(L)) = \rho(\mathfrak{A}(S)) \cap I(R)$.

PROOF. Regard R as a representation space for S under the regular representation and let Z be the center of R . Since $S \circ Z \subseteq Z$ and since every representation of a semi-simple Lie algebra is semi-simple, we can find a vector subspace U of R such that $R = U + Z$, $U \cap Z = (0)$, and $S \circ U \subseteq U$. Now take T in $\rho(\mathfrak{A}(S)) \cap I(R)$ and let r_T be such that $T(r) = r \circ r_T$ for all r in R . Write $r_T = u + z$ with u in U and z in Z . Then $T(r) = r \circ u$ for all r in R . Since $T(S) = (0)$, we have $T(r \circ s) = T(r) \circ s$, i.e. $(r \circ s) \circ u = (r \circ u) \circ s$. Hence by the Jacobi identity, $(s \circ u) \circ r = 0$ for all r in R . Thus $s \circ u$ is in Z so that $s \circ u$ is in $Z \cap U = (0)$, whence $s \circ u = 0$. Therefore u^* is in $\mathfrak{A}(S) \cap I(L)$ and $\rho(u^*) = T$ proving that $\rho(\mathfrak{A}(S)) \cap I(R) \subseteq \rho(\mathfrak{A}(S) \cap I(L))$.

Conversely, take $\rho(x^*)$ in $\rho(\mathfrak{A}(S) \cap I(L))$. x^* in $\mathfrak{A}(S) \cap I(L)$ implies $x^*(s) = 0$ for all s in S , so that if we write $x = s_x + r_x$ with s_x in S and r_x in R , we have $0 = x^*(s) = s \circ s_x + s \circ r_x$ whence $s \circ s_x = r_x \circ s$ for all s in S . Hence $s_x = 0$ so that $x = r_x$ and $x^* = r_x^*$. The latter means that $\rho(x^*)$ is in $I(R)$ so that $\rho(\mathfrak{A}(S) \cap I(L)) \subseteq \rho(\mathfrak{A}(S)) \cap I(R)$ and the proof is complete.

If we remember that ρ is an isomorphism of $\mathfrak{A}(S)$ onto $\rho(\mathfrak{A}(S))$, then the above results include the following lemma:

2.5. LEMMA. $\mathfrak{A}(S)$ splits over $\mathfrak{A}(S) \cap I(L)$ if and only if $\rho(\mathfrak{A}(S))$ splits over $\rho(\mathfrak{A}(S)) \cap I(R)$.

3. Proof of the main theorem. Suppose that E is a complement of $I(R)$ in $D(R)$, i.e. $D(R) = E + I(R)$ and $E \cap I(R) = (0)$. Let \mathfrak{N} be the radical of $D(R)$. Since R is solvable, $I(R)$ is a solvable ideal of $D(R)$ whence $I(R) \subseteq \mathfrak{N}$. Using the notation of §2 and applying Lemma 2.1 we may choose E so that $\rho(S^*) \subseteq E$.

Consider $D(R)$ as a $\rho(S^*)$ module in the regular representation. Clearly, $D(R) = E + I(R)$, direct sum as $\rho(S^*)$ modules (by the choice of E). Let X be the trivial $\rho(S^*)$ submodule of E and Y the trivial $\rho(S^*)$ submodule of $I(R)$. Then $X + Y$ is the trivial $\rho(S^*)$ submodule of $D(R)$. Therefore by Lemma 2.3 we have that $X + Y = \rho(\mathfrak{A}(S))$. Since $Y \subseteq I(R)$, we have $(X + Y) \cap I(R) = X \cap I(R) + Y = Y$. Hence

$Y = \rho(\mathfrak{A}(S)) \cap I(R)$. We show next that X is a subalgebra of $\rho(\mathfrak{A}(S))$. Take T_1, T_2 in X , then

$$\begin{aligned}\rho(s^*) \cdot (T_1 \circ T_2) &= (T_1 \circ T_2) \circ \rho(s^*) \\ &= - (T_2 \circ \rho(s^*)) \circ T_1 - (\rho(s^*) \circ T_1) \circ T_2 \\ &= - (\rho(s^*) \cdot T_2) \circ T_1 + (\rho(s^*) \cdot T_1) \circ T_2 = 0,\end{aligned}$$

so that $T_1 \circ T_2$ is in $\rho(\mathfrak{A}(S))$. But $X \subseteq E$ and E is a subalgebra of $D(R)$ so that $T_1 \circ T_2$ is in E and hence is in X .

Thus X is a complement of $\rho(\mathfrak{A}(S)) \cap I(R) = Y$ in $\rho(\mathfrak{A}(S))$ so that, by Lemma 2.5, $\mathfrak{A}(S)$ splits over $\mathfrak{A}(S) \cap I(L)$ and the theorem follows by Lemma 2.2.

3.1. COROLLARY. *If the radical of L is abelian, then $D(L)$ splits over $I(L)$.*

4. Appendix. We now give an example of a nilpotent Lie algebra L which is such that $D(L)$ does not split over $I(L)$.⁵

Let L have a basis x_1, x_2, x_3, x_4 over a ground field K where (x_2, x_3, x_4) is abelian and $x_1 \circ x_2 = x_4, x_1 \circ x_3 = 0, x_1 \circ x_4 = 0$. Clearly $L \circ L = (x_4)$ and the center of L is (x_3, x_4) . $I(L)$ consists of derivations of the form $k_1 x_1^* + k_2 x_2^*$ and we note that $(k_1 x_1^* + k_2 x_2^*)(x_1) = k_2 x_4$, $(k_1 x_1^* + k_2 x_2^*)(x_2) = -k_1 x_4$. Thus if D_1 is any derivation of L we may add an inner derivation to it so that the resulting derivation has the form:

$$\begin{aligned}D(x_1) &= ax_1 + bx_2 + cx_3, D(x_2) = dx_1 + ex_2 + fx_3, \\ D(x_3) &= gx_3 + hx_4, \quad D(x_4) = (a+e)x_4.\end{aligned}$$

Note that no such derivation can be in $I(L)$.

Suppose now that $D(L) = E + I(L)$ is a splitting of $D(L)$ over $I(L)$ and consider the derivations D_1, D_2 such that $D_1(x_1) = x_3, D_1(x_i) = 0$ if $i \neq 1$; $D_2(x_3) = x_4, D_2(x_i) = 0$ if $i \neq 3$. Then $D_1 \circ D_2 = x_2^*$. Write $D_1 = e_1 + r_1^*, D_2 = e_2 + r_2^*$ with e_1, e_2 in E , and $r_1 = k_1 x_1 + k_2 x_2, r_2 = k'_1 x_1 + k'_2 x_2$. Then

$$\begin{aligned}e_1 \circ e_2 &= (D_1 - r_1^*) \circ (D_2 - r_2^*) = D_1 \circ D_2 - D_1 \circ r_2^* \\ &\quad + D_2 \circ r_1^* - r_1^* \circ r_2^* \\ &= x_2^* + (D_1(r_2))^* - (D_2(r_1))^* - (r_1 \circ r_2)^* = x_2^*,\end{aligned}$$

i.e. $e_1 \circ e_2$ is in $E \cap I(L)$ so that $e_1 \circ e_2 = 0 = x_2^*$, which is a contradiction.

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⁵ This example was communicated to me in a letter by Professor Hochschild.