A REMARK ON ZETA FUNCTIONS

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1. Let \( s = \sigma + it \), \( \xi(s) = \sum_n n^{-s} \) (\( \sigma > 1 \)), \( \omega(x) = \sum_{n=1}^\infty e^{-n^2 x} \) (\( x > 0 \)),

\[
\xi(s) = \pi^{-s/2} \Gamma(s/2) \xi(s) = \int_0^\infty \omega(x) x^{s/2-1} dx \quad (\sigma > 1).
\]

Then \( s(s-1)\xi(s) \) is well known to be an entire function, and its zeros are identical with the nontrivial zeros of \( \xi(s) \), i.e. with those lying in the strip \( 0 < \sigma < 1 \). Furthermore let

\[
w = u + iv, w \neq 0; |\arg w|, |\arg 1/w|, |\arg (x + w)| \leq \pi/2; s \neq 0, 1,
\]

be fixed;

\[
F_u(w) = \int_0^\infty \omega(x) (x + w)^{s/2-1} dx - \frac{w^{(s-1)/2}}{1 - s}.
\]

Then \( F_u(w) \) is an analytic function of \( w \) for \( u > 0 \), since

\[
|\omega(x + w)| \leq x^{-1/2} e^{1-x} \quad (x > 0);
\]

its limit function \( F_u(\bar{w}) \) exists for any \( v \geq 0 \) by the Lebesgue convergence theorem. Now we can deduce that

\[
F_u(w) + F_{1-u}(1/w) = \xi(s) \quad (u \geq 0, w \neq 0).
\]

For \( v = 0 \) this reduces to the, possibly known, equation

\[
\xi(s) = \int_u^\infty \omega(x) x^{s/2-1} dx + \int_{1/u}^\infty \omega(x) x^{-(1+s)/2} dx - \frac{u^{(s-1)/2}}{1 - s} - \frac{u^{s/2}}{s}.
\]

Hence (1.3) hold by analytic continuation. Clearly \( F_u(w) \rightarrow \xi(s) (w \rightarrow 0; 0 < \sigma < 1) \).

Again (1.3) takes simple forms for \( w = i \) and \( w = 2i \):

\[
x(s) = \int_0^\infty \lambda(x) (x + i)^{s/2-1} dx
\]

\[
+ \int_0^\infty \lambda(x) (x - i)^{-(s+1)/2} dx - \frac{e^{i\pi(s-1)/4}}{1 - s} - \frac{e^{i\pi s/4}}{s};
\]

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1 For \( u = 1 \) this is the classical equation due to Riemann. E.g. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Leipzig and Berlin, 1909, §70; by a similar argument (1.3a) is deduced.
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\( \xi(s) = \int_0^\infty \omega(x)(x + 2i)^{s-1/2} dx \)

\[ (1.3c) \quad \xi(s) = \int_0^\infty \left\{ i\omega(x) + (1 - i)\omega(4x) \right\} \left( x - \frac{i}{2} \right)^{-s-1/2} dx 
- \frac{(2e^{i\pi/2})^{(s-1)/2}}{1 - s} \frac{(2e^{i\pi/2})^{s/2}}{s}; \]

where \( \lambda(x) = \sum_{n=1}^\infty (-1)^n e^{-\pi n x} \). By the formula \(|\Gamma(s)| e^{\pi t/2} t^{1/2-s}\to\text{constant} (0 < t \to \infty)\) the Lindelöf hypothesis is equivalent to the statement
\[
\Re \left\{ \int_0^\infty \lambda(x)(x + i)^{-3/4 + it/2} dx \right\} = O(t^{-1/4 + e^{-\pi t/4}})
\]
\( \epsilon > 0 \text{ fixed; } 0 < t \to \infty \).

2. The following theorems, easily derived from (1.3), give criteria for the nontrivial zeros of \( \xi(s) \):

Theorem 1. A given point \( s \) is a nontrivial zero of \( \xi(s) \) if and only if, for \( w \neq 0 \) with \( u \geq 0 \), \( F_s(w) \) satisfies the functional equation
\[
F_s(w) = - F_{1-s}(1/w).
\]

Theorem 2. Let \( 0 < \sigma < 1 \). Then \( s \) is a zero of \( \xi(s) \) if and only if
\[
w^{(s-1)/2} \int_0^\infty (x + w)^{-\sigma} \left\{ \frac{1}{2} (x + w)^{-1/2} - \omega(x + w) \right\} dx
\]
\[ \to \frac{1}{s - 1} \quad (u \geq 0; \ |w| \to 0), \]

or, for any fixed \( a > 0 \) (for instance, for \( a = 1 \)),
\[
\int_0^a x^{s/2-1} \left\{ \frac{1}{2} x^{-1/2} - \rho^{1/2} \omega(x\rho) \right\} dx \to \frac{a^{(s-1)/2}}{s - 1} \quad (0 < \rho \to \infty).
\]

If \( s \) is not a zero, the moduli of the terms on the left of (i) and (ii) tend to infinity.

Remark. For fixed \( s \) \( (0 < \sigma < 1) \), \( F_s(w) \sim w^{(s-1)/2}(s-1)^{-1} \to 0 \) \( (|w| \to \infty) \); \( F_s(w) \) is bounded and uniformly continuous \( (u \geq 0) \); \( F_s(w) = (2\pi)^{-1} \int_{-\infty}^\infty d\omega \omega \omega(\omega - i\sigma)^{-1} \) \( (u > 0) \); and \( \xi(s) = i/\pi \) PV.

3 The assertion, still unproved, that \( \xi(1/2 + it) = O(t^\epsilon) \) \((\epsilon > 0; 0 < t \to \infty)\). This is known to be equivalent to \( \int_{-\infty}^\infty x^{-3/4 + it} \lambda(x) dx = O(t^{-1/4 + e^{-\pi t/4}}) \).

3 PV. \( \int_{-\infty}^\infty = \lim_{a \to \infty} (\int_{-a}^0 + \int_a^\infty) \) is the "principal value" of the integral. The above representations of \( F_s(w) \) and \( \xi(s) \) by integrals follow from the theory of the Hille-Tamarkin class \( \mathcal{S}_p \); see Fund. Math. vol. 25 (1935) pp. 329-352.
In a recent paper T. M. Apostol has investigated the functional equation of the generalized zeta function \( \phi(s, a, b) = \sum_{n=0}^{\infty} e^{2\pi inb}(n + a)^{-s} \), due to Lerch, for the case \( b \to 1 \) (\( 0 < a < 1 \), \( 0 < b < 1 \)). The problem can be considerably simplified and, incidentally, generalized. Replace \( \phi(s, a, b) \) by \( \xi(s, a, b) \) and introduce \( Z_1(s, a, b) \), \( Z_2(s, a, b) \), where \( a \) and \( b \) are any real numbers,

\[
\xi(s, a, b) = \sum_{n>-a} e^{2\pi inb}(n + a)^{-s},
\]

\[
Z_1(s, a, b) = \sum_{n=-\infty, n+a>0} e^{2\pi inb} \left| n + a \right|^{-s},
\]

\[
Z_2(s, a, b) = \sum_{n=-\infty, n+a>0} \frac{e^{2\pi inb}(n + a)}{\left| n + a \right|^{s+1}} \quad (s < 1).
\]

Obviously

\[
2\xi(s, a, b) = Z_1(s, a, b) + Z_2(s, a, b);
\]

\[
2\xi(s, -a, -b) = Z_1(s, a, b) - Z_2(s, a, b),
\]

and we obtain the functional equations

\[
e^{2\pi inb} \chi(s + k - 1) Z_4(s, a, b) = i^{k-1} \chi(k - s) Z_4(1 - s, b, -a)
\]

\[
\frac{(2\pi)^s}{\Gamma(s)} e^{2\pi isb} \xi(1 - s, a, b)
\]

\[
= e^{\pi is/2} \xi(s, b, -a) + e^{-\pi is/2} \xi(s, -b, a),
\]

where \( \chi(s) = \pi^{-s/2} \Gamma(s/2) \); \( a, b \) real. The equation (3.2), known in special cases, is deduced from well known formulae on theta series, by the classical method; while, by (3.1), (3.3) is a corollary of it.

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References:


5 E.g. H. Kober, J. Reine Angew. Math. vol. 174 (1936) pp. 206–225, §4. Again the equation (3.2) for \( Z_4(s, a, b) \) is deduced by Apostol in the special case \( 0 < a < 1 \), Pacific Journal of Mathematics vol. 1 (1951) pp. 161–167. For his function \( \Lambda(x, a, s) \), defined for \( 0 < a < 1 \) and treated by the classical method (see pp. 161–163), reduces to \( Z_4(s, a, x) \), etc. as is easily shown.

6 I.e. \( \theta(x, a, b) = e^{2\pi inb} x^{-1} \theta(x^{-1}, b, -a) \) and the formula gained from this by differentiation with respect to \( a \); where \( \theta(x, a, b) = \sum_{-\infty}^{\infty} \exp \left\{ -\pi x(n+a)^2 + 2\pi inb \right\} = \theta(x, -a, -b) \).