NOTE ON SOME PARTITION IDENTITIES

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1. Introduction. In a recent paper, Newman [4] states the formulas

\[(1.1) \quad \sum_{n=0}^{\infty} p_3(11m + 10)x^n = \sum_{n=1}^{\infty} (1 - x^{11n})^2,\]

\[(1.2) \quad \sum_{n=0}^{\infty} p_4(11m + 20)x^n = -11 \prod_{n=1}^{\infty} (1 - x^{11n})^4,\]

\[(1.3) \quad \sum_{n=0}^{\infty} p_5(17m + 24)x^n = - \prod_{n=1}^{\infty} (1 - x^{17n})^2,\]

\[(1.4) \quad \sum_{n=0}^{\infty} p_6(31m + 240)x^n = 961 \prod_{n=1}^{\infty} (1 - x^{31n})^4,\]

where

\[\prod_{n=1}^{\infty} (1 - x^n)^k = \sum_{m=0}^{\infty} p_k(m)x^m.\]

We wish to point out that results of this kind can be obtained in a very elementary way, namely, by using a method employed by Ramanujan in proving the formula \(p(5m+4) \equiv 0 \pmod{5}\) (see for example [2, p. 87]). We shall prove the following formulas. Let \(r\) be prime. If \(r \equiv 3 \pmod{4}\), \(r > 3\), then

\[(1.5) \quad \sum_{m=0}^{\infty} p_3(3r + r_0)x^m = \prod_{n=1}^{\infty} (1 - x^{rn})^2,\]

where \(r_0 = (r^2 - 1)/12\).

If \(r \equiv 3 \pmod{4}\), \(r \geq 3\), then

\[(1.6) \quad \sum_{m=0}^{\infty} p_4(3r + r_1)x^m = r^2 \prod_{n=1}^{\infty} (1 - x^{rn})^4,\]

where \(r_1 = (r^2 - 1)/4\).

If \(r \equiv 5 \pmod{6}\), then

\[(1.7) \quad \sum_{m=0}^{\infty} p_5(3r + r_2)x^m = -r \prod_{n=1}^{\infty} (1 - x^{rn})^4,\]

where \(r_2 = (r^2 - 1)/6\).

Received by the editors October 27, 1952.
If \( r \equiv 5 \pmod{12} \), then

\[
(1.8) \quad \sum_{m=0}^{\infty} p_2(rm + r_0)x^m = -\prod_{n=1}^{\infty} (1 - x^n)^2,
\]

where \( r_0 = (r^2 - 1)/12 \).

It is clear that (1.1) is contained in (1.5), (1.2) in (1.7), (1.3) in (1.8), (1.4) in (1.6); the case \( r = 5 \) of (1.7) occurs in [3]. We also remark that (1.5), \ldots, (1.8) can be put in somewhat sharper form; for example in place of (1.5) we can state

\[
\sum_{m=0}^{\infty} p_2(r^2m + r_0)x^m = \prod_{n=1}^{\infty} (1 - x^n)^2 = \sum_{m=0}^{\infty} p_2(m)x^m.
\]

In other words

\[
p_2(r^2m + r_0) = p_2(m); \quad p_2(rm + r_0) = 0 \quad \text{for} \quad r \nmid m.
\]

Similar results hold for the other functions.

2. Proof of (1.5). By Euler's formula

\[
(2.1) \quad x^s \prod_{n=1}^{\infty} (1 - x^n)^2 = \sum_{h, k=\infty}^{\infty} (-1)^{h+k} x^{e + h(2k+1)/2 + k(3h+1)/2},
\]

where \( s \) is to be assigned. The exponent on the right is divisible by \( r \) provided

\[
(2.2) \quad (6h + 1)^2 + (6k + 1)^2 + 2(12s - 1) \equiv 0 \pmod{r}.
\]

If we take \( s \) as the least positive integer such that \( 12s \equiv 1 \pmod{r} \), then by the hypothesis on \( r \) it is clear that (2.2) implies \( r|6h+1, \quad r|6k+1 \). Thus with a little manipulation (2.1) yields

\[
\sum_{m=0}^{\infty} p_2(rm + r - s)x^m = x^s \prod_{n=1}^{\infty} (1 - x^n)^2,
\]

where

\[
(2.3) \quad e = \frac{12s - 1}{12r} + \frac{r}{12} - 1.
\]

Since

\[
re + r - s = \frac{12s - 1}{12} + \frac{r^2}{12} - s = \frac{r^2 - 1}{12},
\]

(1.5) follows at once.
3. Proof of (1.6). Using Jacobi’s formula we have

\[ x^s \prod_{n=1}^{\infty} (1 - x^n)^6 \]

\[= \sum_{h,k=0}^{\infty} (-1)^{h+k}(2h+1)(2k+1)x^{s+h(k+1)} \frac{1}{2+h(k+1)/2}. \]

The exponent on the right is divisible by \( r \) provided

\[(2h+1)^2 + (2k+1)^2 + 2(4s-1) \equiv 0 \pmod{r}. \]

If we choose \( s \) as the least positive integer such that \( 4s \equiv 1 \pmod{r} \), (3.2) implies \( r \mid 2h+1, r \mid 2k+1 \). Thus, very much as above, (3.1) yields

\[ \sum_{m=0}^{\infty} p_s(rm + r - s)x^m = r^2 x^e \prod_{n=1}^{\infty} (1 - x^n)^6, \]

where

\[ e = \frac{8s-1}{8r} + \frac{r}{4} - 1. \]

Since

\[ re + r - s = \frac{8s-1}{8} + \frac{r^2}{4} - s = \frac{r^2 - 1}{4}, \]

(1.6) follows at once.

4. Proof of (1.7). Using Euler’s and Jacobi’s formula we have

\[ x^s \prod_{n=1}^{\infty} (1 - x^n)^4 = \frac{1}{2} \sum_{h,k=0}^{\infty} (-1)^{h+k}(2h+1)x^{s+h(k+1)}/2+h(k+1)/2. \]

The exponent on the right is divisible by \( r \) provided

\[(6h+1)^2 + 3(2k+1)^2 + 4(6s-1) \equiv 0 \pmod{r}. \]

We choose \( s \) as the least positive integer such that \( 6s \equiv 1 \pmod{r} \). Since \(-3\) is a quadratic nonresidue of \( r \), it follows from (4.2) that \( r \mid 6h+1, r \mid 2k+1 \). A little attention must now be paid to the sign in the right member of (4.1). We find without much trouble that (4.1) implies

\[ \sum_{m=0}^{\infty} p_s(rm + r - s)x^m = -r x^e \prod_{n=1}^{\infty} (1 - x^n)^4, \]

where
\[ e = \frac{s - 1}{6r} + \frac{r}{6} - 1. \]

Since
\[ re + r - s = \frac{s - 1}{6} + \frac{r^2}{6} - s = \frac{r^2 - 1}{6}, \]
it is evident that (4.3) reduces to (1.7).

5. Proof of (1.8). We return to (2.1) and (2.2). Since \( r \equiv 1 \pmod{4} \) we can no longer assert that \( r \mid 6k + 1, r \mid 6k + 1 \), but only that \( (6k + 1)^2 + (6k + 1)^2 \equiv 0 \pmod{p} \). Changing the notation slightly, consider
\[
\begin{align*}
5h' & \equiv 3h - 4k \equiv -1, & h' & \equiv 1 \pmod{6}, \\
5k' & \equiv -4h + 3k = -1, & k' & \equiv 1.
\end{align*}
\]
On the other hand (5.2) implies
\[
\begin{align*}
5h' & \equiv 3h - 4k \equiv -1, & h' & \equiv 1 \pmod{6}, \\
5k' & \equiv -4h + 3k = -1, & k' & \equiv 1.
\end{align*}
\]
It follows that the terms in the right member of (2.1) corresponding to \((h, k)\) and \((h', k')\) cancel.

Next, if \( b \equiv -2 \pmod{6} \), we change all signs in the right members of (5.2). The details are much as before; in particular (5.3) becomes \( h' \equiv h, k' \equiv -k \pmod{4} \). Thus once again corresponding terms cancel.

Now consider a pair \((h, k)\) with \( h^2 + k^2 = m \), where \( m \) is fixed, \( r \mid m \), \( h \equiv k \equiv 1 \pmod{6} \). Suppose first \( r \mid h \). Then if \( r \mid h' \), it is clear from the above that the corresponding terms in (2.1) cancel. On the other hand, when \( r \nmid h \), then it follows from the above discussion that we can simultaneously consider the correspondence (5.2) together with the second correspondence \((b \equiv -2)\). In other words we have in this case \((r \mid h)\) a \((2, 1)\) correspondence. Returning to (2.1) we see that
\[
\sum_{m=0}^{\infty} p_{\pi}(rm + r - s) = -x^r \prod_{n=1}^{\infty} (1 - x^{rn})^2,
\]
where $e$ is determined by (2.3). The proof of (1.8) is now completed in exactly the same way as in (1.5).

6. **Another formula.** Newman also states the formula

\begin{equation}
\sum_{n=0}^{\infty} p_6(5m)x^m = \prod_{n=1}^{\infty} (1 - x^n)^6(1 - x^{5n})^{-1},
\end{equation}

which he notes had been found (but not published) by D. H. Lehmer. It may be of interest to point out that (6.1) can be obtained easily from the identity.

\begin{equation}
\prod_{n=1}^{\infty} \frac{(1 - x^n)^6}{1 - x^{5n}} = 1 - 5 \sum_{m=1}^{\infty} \left( \frac{m}{5} \right) \frac{x^m}{1 - x^m}.
\end{equation}

The formula (6.2) is due to Ramanujan; Bailey [1] showed recently that it is a consequence of well-known formulas for the Weierstrass elliptic functions.

Since the right member of (6.2) equals

\[ 1 - 5 \sum_{m,r=1}^{\infty} \left( \frac{m}{5} \right) m x^{m^r}, \]

it follows that

\begin{align*}
\sum_{m=0}^{\infty} p_6(5m)x^{5m} \prod_{n=1}^{\infty} (1 - x^{5n})^{-1} &= 1 - 5 \sum_{m=1}^{\infty} \left( \frac{m}{5} \right) \frac{x^{5m}}{1 - x^{5m}} \\
&= \prod_{n=1}^{\infty} (1 - x^{5n})^{6}(1 - x^{25n})^{-1}.
\end{align*}

Replacing $x^a$ by $x$ we get (6.1).

**References**


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