A CARDINAL NUMBER ASSOCIATED WITH A FAMILY OF SETS

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Let $U$ be a family of nonempty subsets of an abstract set $R$, partially ordered by set inclusion. The smallest cardinal number which is the power of the union of a maximal family of incomparable elements of $U$ shall be defined as the "maximal density" of $U$ ($\text{md}(U)$). The smallest cardinal number which is the md $(V)$ of some coinitial subfamily $V$ of $U$ shall be defined as the "containing maximal density" of $U$ ($\text{cmd}(U)$). The principal result of this paper is Theorem 2, which states that $\prod_{t \in a} \text{cmd} (U_t) = \text{cmd} (\prod_{t \in a} U_t)$.

Before turning to our main result we consider the containing maximal density of a ramified family of sets.

THEOREM 1. If $U$ is a ramified family of sets, then the maximal density of $U$ equals the containing maximal density of $U$.

Proof. Let $V = \{E\}$ be a coinitial subfamily of $U = \{D\}$, such that $\text{md}(V) = \text{cmd}(U)$. Let $M = \{H\}$ be a maximal family of incomparable elements of $V$ for which $\text{md}(V) = \rho(M)$. We shall now show that $M$ is a maximal family of incomparable elements of $U$. Let $D$ be any element in $U$, and $E$ any element of $V$ which is a subset of $D$. The element $E$ certainly exists since $F$ is a coinitial subset of $U$. The family $M$ being a maximal family of incomparable elements of $V$, there exists an element $H$ in $M$ which is comparable with $E$. If $H$ is a subset of $E$, then $H$ is also a subset of $D$. Suppose that $E$ is a subset of $H$. Since $U$ is ramified, it follows that the two elements $D$ and $H$ are comparable. Consequently each element of $U$ is comparable with some element in $M$. Thus $M$ is a maximal family of incomparable elements in $U$. Therefore

$$\text{md}(U) \leq \text{md}(V) = \text{cmd}(U) \leq \text{md}(U).$$

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2 In the sequel $U$ and $U^t$ will be a family of subsets of the sets $R$ and $R^t$ respectively.

3 The family of sets $U$ is a ramified family of sets if, for each element $E$ in $U$, the family of sets $\{D \supseteq E, D \subseteq U\}$ is monotone.

4 Let $Y$ be a family of sets. By $\rho(Y)$ is meant the power of the set which is the set union of the elements in $Y$. If $Y$ consists of only one set, say $A$, then $\rho(Y)$ is the power of $A$. In this case, $\rho(A)$ is also used, i.e., $\rho(A) = \rho(Y)$.
Thus \( \text{md}(U) = \text{cmd}(U) \).

In preparation for Theorem 2, two lemmas are needed.

**Lemma 1.** The containing maximal density of \( U \) is the smallest cardinal number equal to \( \rho(V) \), for some coinitial subfamily \( V \) of \( U \).

**Proof.** Let \( V = \{ E \} \) be any coinitial subfamily of \( U \) for which \( \text{cmd}(U) = \text{md}(V) \). Let \( M = \{ E_\alpha \} \) be a maximal family of incomparable elements of \( V \) such that \( \rho(M) = \text{md}(V) \). Now the subfamily of \( V \),

\[
Y = \{ E \mid E \subseteq E_\alpha, E \in V, E_\alpha \in M, \text{ for some } \nu \},
\]

is coinitial in \( V \). Thus \( \rho(Y) \leq \text{md}(V) = \text{cmd}(U) \). Now let \( \rho(Z) \) be the smallest cardinal number for some coinitial subfamily \( Z \) of \( U \). Clearly \( \text{cmd}(U) \leq \text{md}(Z) \leq \rho(Z) \). This completes the proof.

**Lemma 2.** To each family \( U = \{ D \} \), there corresponds a coinitial subfamily \( V = \{ E \} \), and a subfamily, \( Y = \{ D_\nu \mid \nu < \delta \} \) of \( V \), which have the following properties:

1. Each element of \( V \) is \( p \)-homogeneous;
2. \( \{ E \mid E \subseteq D_\xi, E \in V \} \cap \{ E \mid E \subseteq D_\nu, E \in V \} = \emptyset \) for \( \xi \neq \nu \);
3. If \( \{ G_\xi \mid \xi < \delta \} \) is any subfamily of \( V \) in which \( G_\xi \) is a subset of \( D_\xi \) for each \( \xi \), then

\[
\rho(\bigcup_{\xi < \delta} G_\xi - U D_\xi) = \rho(\bigcup_{\xi < \delta} D_\xi)
\]

for \( \nu < \delta \).

4. \( \rho(U < < D_\nu - \bigcup_{\xi < \nu} D_\xi) \geq \text{cmd}(V) \).

**Proof.** If \( V \) is a coinitial subfamily of \( U \) such that \( \text{md}(V) = \text{cmd}(U) \), and

\[
Z = \{ E \mid E \in V, E \text{ is } p \text{-homogeneous} \},
\]

then \( Z \) is a coinitial subfamily of \( U \) for which \( \text{md}(Z) = \text{cmd}(U) \). In order to simplify the notation, it is assumed that \( U \) has the two properties of \( Z \), i.e., (a) \( \text{md}(U) = \text{cmd}(U) \), and (b) each element \( D \) in \( U \) is \( p \)-homogeneous. Well order the elements of \( U \), \( D_0 \) being the first element. Suppose that the family of sets \( \{ D_\xi \mid \xi < \lambda \} \) has already been defined. Denote by \( D_\nu \) the first element \( D_\nu \) in \( U \) which satisfies the following two conditions:

1. If \( D \) is a subset of \( D_\nu \), where \( D \) is in \( U \), then \( D \) is not a subset of \( \bigcup_{\xi < \nu} D_\xi \).

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\(^6\) An element \( E \) in \( V \) is \( p \)-homogeneous if \( \rho(B) = \rho(E) \) for each element \( B \) in \( V \) which is a subset of \( E \). See Erdös and Tarski, *On families of mutually exclusive sets*, Ann. of Math. vol. 44 (1943) pp. 315–329.
(d) if \( D \) is a subset of \( D_\bullet \), where \( D \) is in \( U \), then \( \rho(D - \bigcup_{\tau<\alpha} D_\tau) = \rho(D_\bullet - \bigcup_{\tau<\alpha} D_\tau) \). If the set of elements satisfying (c) is nonempty, then the element \( D_\lambda \) certainly exists. Let \( \{D_\xi|\xi<\delta\} \) be a maximal family obtained in this way.

Let \( V = \{E\} \) be the subfamily of \( U \),

\[
V = \left\{D \mid D \subseteq \bigcup_{\tau<\alpha} D_\tau, D \in U\right\}.
\]

Conditions (1), (2), (3), and (4) are automatically satisfied. It is necessary to show only that \( V \) is a cofinal subfamily of \( U \). Suppose that \( D_\bullet \) is an element of \( U \) which contains no element of \( V \). Then the element \( D_\bullet \) satisfies condition (c). This implies that the family \( \{D_\xi|\xi<\delta\} \) is not maximal. From this contradiction we obtain our conclusion.

We now prove our main result.

**Theorem 2.** \( \text{cmd} \left( \prod_{\xi<\alpha} U^\xi \right) = \prod_{\xi<\alpha} \text{cmd} \left( U^\xi \right) \), where by \( \prod_{\xi<\alpha} U^\xi \) is meant the cartesian product of the \( U^\xi \).

**Proof.** To each family \( U^\xi \), associate a family of sets, \( Y^\xi = \{D_\gamma|\gamma<\delta^\xi\} \), which satisfies the conclusions of Lemma 2. Let \( U \) be a cofinal subfamily of \( \prod_{\xi} U^\xi \) for which \( \rho(U) = \text{cmd} (U) = \text{cmd} (U) \).

In the following, \( \beta, \gamma, \xi \) and \( \xi \) denote indices that range over specified sets of ordinal numbers, whereas \( \mu \) and \( \nu \) denote indices that range over the class of all \( \alpha \)-sequences \( \sigma = \{\sigma_\xi\}, \xi<\alpha \), where \( \sigma_\xi<\delta_\xi \) for each \( \xi \). The notation \( \mu<\nu \) refers to the partial ordering defined by

\[
\mu < \nu \text{ if and only if } \begin{cases} 
\mu_\xi \leq \nu_\xi & \text{for all } \xi < \alpha, \\
\mu_\xi < \nu_\xi & \text{for some } \xi < \alpha.
\end{cases}
\]

For each \( \nu \), \( \prod_{\xi} D^\xi_{\nu_\xi} \) belongs to \( \prod_{\xi} U^\xi \). Since \( U \) is a cofinal subfamily of \( \prod_{\xi} U^\xi \), there exists an element \( D_\nu \) in \( U \) such that \( D_\nu \subseteq \prod_{\xi} D^\xi_{\nu_\xi} \). \( D_\nu \) is of the form \( D_\nu = \prod_{\xi} E^\xi_{\nu_\xi} \), where for each \( \xi < \alpha \), \( E^\xi_{\nu_\xi} \) is an element of \( U^\xi \) and \( E^\xi_{\nu_\xi} \subseteq D^\xi_{\nu_\xi} \). It now follows that

\[
\text{cmd} \left( \prod_{\xi} U^\xi \right) = \rho(U) \geq \rho \left( \bigcup_{\nu} D_\nu \right) \\
\geq \rho \left[ \bigcup_{\nu} \left( D_\nu - \bigcup_{\mu<\nu} D_\mu \right) \right] \geq \rho \left( \bigcup_{\nu} \left( D_\nu - \bigcup_{\mu<\nu} \prod_{\xi} D^\xi_{\nu_\xi} \right) \right).
\]

Since each two terms inside the last bracket are disjoint,
\[
\begin{align*}
\rho \left[ \bigcup_{\nu < \tau} \prod_{\xi} D^\xi_{\nu} \right] \\
= \sum_{\nu} \rho \left( D^\tau_{\nu} \right) - \bigcup_{\nu < \tau} \prod_{\xi} D^\xi_{\nu} \\
= \sum_{\nu} \rho \left( \prod_{\xi} E^\xi_{\nu} \right) - \bigcup_{\nu < \tau} \prod_{\xi} D^\xi_{\nu} ,
\end{align*}
\]

which, since \( \prod_{\xi} (E^\xi_{\nu} - \bigcup_{\nu < \tau} D^\xi_{\nu}) \subseteq (\prod_{\xi} E^\xi_{\nu} - \bigcup_{\nu < \tau} \prod_{\xi} D^\xi_{\nu}) \), is

\[
\geq \sum_{\nu} \rho \left[ \prod_{\xi} \left( E^\xi_{\nu} - \bigcup_{\nu < \tau} D^\xi_{\nu} \right) \right] \\
= \sum_{\nu} \prod_{\xi} \rho \left( E^\xi_{\nu} - \bigcup_{\nu < \tau} D^\xi_{\nu} \right) \\
= \sum_{\nu} \prod_{\xi} \rho \left( D^\xi_{\nu} - \bigcup_{\nu < \tau} D^\xi_{\nu} \right) \quad \text{(by condition 3)} \\
= \prod_{\xi} \sum_{\nu < \tau} \rho \left( D^\xi_{\nu} - \bigcup_{\nu < \tau} D^\xi_{\nu} \right) \quad \text{(by the distributive law)} \\
= \prod_{\xi} \rho \left[ \bigcup_{\nu < \tau} \left( D^\xi_{\nu} - \bigcup_{\nu < \tau} D^\xi_{\nu} \right) \right] \\
\geq \prod_{\xi} \rho \left( \bigcup_{\nu < \tau} \left( D^\xi_{\nu} - \bigcup_{\nu < \tau} D^\xi_{\nu} \right) \right),
\]

Thus \( \rho \left( \prod_{\xi} U^\xi \right) \geq \prod_{\xi} \rho \left( U^\xi \right) \). Since the reverse inequality is obviously true, it follows that \( \prod_{\xi} \rho \left( U^\xi \right) = \rho \left( \prod_{\xi} U^\xi \right) \).

For each \( \xi < \mu \) let \( U^\xi \) be a family of infinite subsets of a set \( \mathcal{R}^\xi \). By \( \prod_{\xi < \mu} U^\xi \) we shall mean the family of sets

\[
\{ G | G = \prod_{\xi} \mathcal{H}^\xi , \text{where } \mathcal{H}^\xi \subseteq U^\xi , \text{or } \mathcal{H}^\xi = \mathcal{R}^\xi , \text{and all but a finite number of the } \mathcal{H}^\xi \text{ are } \mathcal{R}^\xi \}.
\]

**Theorem 3.** \( \rho \left( \prod_{\xi} U^\xi \right) \) is the smallest cardinal number, call it \( \aleph_\beta \), which is in the set of cardinal numbers \( \{ \aleph_\alpha | \aleph_\alpha = \rho \left( \prod_{\xi} U^\xi \right) \} \times U^{\alpha(1)} \times \cdots \times U^{\alpha(\omega)} \times \prod_{\xi < \omega} \rho (\mathcal{R}^\xi) ; n < \omega \} \).

**Proof.** If \( \mathcal{V}^\xi \) is a coinitial subfamily of \( U^\xi \), then \( \rho \left( \prod_{\xi} U^\xi \right) = \rho \left( \prod_{\xi} \mathcal{V}^\xi \right) \). Thus, no generality is lost in assuming that for each \( \xi \)

\[
\rho (U^\xi) = \rho \left( U^\xi \right) = \rho \left( U^\xi \right).
\]

Furthermore, it may be assumed that
\[ \mathfrak{K}_\beta = \text{cmd} \left( U^0 \times \cdots \times U^n \right) \cdot \prod_{\ell > n} \mathfrak{p}(R^\ell). \]

To prove the theorem, it is sufficient to show that \( \mathfrak{K}_\beta \leq \text{cmd} \left( \prod' U^\ell \right) \). Let \( U \) be a coinitial subfamily of \( \prod' U^\ell \) for which \( \mathfrak{p}(U) = \text{md} \left( U^\ell \right) = \text{cmd} \left( \prod' U^\ell \right) \), and let \( D \) be any element of \( U \). To simplify the notation, suppose that

\[ D = D^0 \times \cdots \times D^{n+m} \times \prod_{\ell > n+m} R^\ell. \]

Then

\[ \text{cmd} \left( \prod' U^\ell \right) = \text{md} \left( U^\ell \right) = \mathfrak{p}(U) \geq \mathfrak{p}(D) \]
\[ = \mathfrak{p}(D^0 \times D^1 \times \cdots \times D^{n+m}) \cdot \prod_{\ell > n+m} \mathfrak{p}(R^\ell) \]
\[ \geq \prod_{\ell > n+m} \mathfrak{p}(R^\ell). \]

Also,

\[ \text{cmd} \left( \prod' U^\ell \right) \geq \text{cmd} \left( U^0 \times \cdots \times U^{n+m} \right) \]
\[ = \prod_{\ell \leq n+m} \text{cmd} \left( U^\ell \right) \quad \text{(Theorem 2)} \]
\[ = \prod_{\ell \leq n+m} \mathfrak{p}(U^\ell) = \mathfrak{p} \left( \prod_{\ell \leq n+m} U^\ell \right). \]

Combining our results we get

\[ \mathfrak{K}_\beta \leq \mathfrak{p} \left( \prod_{\ell \leq n+m} U^\ell \right) \cdot \mathfrak{p} \left( \prod_{\ell > n+m} R^\ell \right) \]
\[ \leq \text{cmd} \left( \prod_{\ell \leq n+m} U^\ell \right) \cdot \text{cmd} \left( \prod_{\ell} U^\ell \right) \]
\[ = \text{cmd} \left( \prod' U^\ell \right), \]

the last equality resulting from the fact that as each element of \( U^\ell \) is infinite, \( \text{cmd} \left( \prod' U^\ell \right) \) is infinite.

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