ON EULER METHODS OF SUMMABILITY FOR DOUBLE SERIES
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The two qth order Euler transforms of the sequence $A_n$

$$A^q_n = (q + 1)^{-n-1} \sum_{k=0}^{n} \binom{n+1}{k+1} q^{n-k} A_k$$

and

$$B^q_n = (q + 1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} A_k$$

are equivalent for $q \geq 0$ in the sense that if either has a limit as $n \to \infty$ the other has the same limit [1, p. 180].\footnote{1 Numbers in brackets refer to the references at the end of the paper.}

For double sequences the corresponding transforms are

(1) $A^q_{mn} = (q + 1)^{-m-n-2} \sum_{h,k=0}^{m,n} \binom{m+1}{h+1} \binom{n+1}{k+1} q^{m+n-h-k} A_{hk}$,

(2) $B^q_{mn} = (q + 1)^{-m-n} \sum_{h,k=0}^{m,n} \binom{m}{h} \binom{n}{k} q^{m+n-h-k} A_{hk}$.

This paper is concerned with two theorems regarding these transforms. Throughout the discussion $q \geq 0$.

Theorem 1. If $A^q_{mn}$ has a limit as $m, n \to \infty$, then $B^q_{mn}$ has that same limit and if $B^q_{mn}$ has a limit and is bounded, then $A^q_{mn}$ has that same limit but there do exist sequences for which $B^q_{mn}$ has a limit but for which $\lim_{m,n \to \infty} A^q_{mn}$ does not exist for any $q \geq 0$.

The relation

(3) $B^q_{mn} = q^2 A^q_{m-1,n-1} - q(q + 1)(A^q_{m,n-1} + A^q_{m-1,n}) + (q + 1)^3 A^q_{mn}$

may be verified by substitution from (1) into the right-hand side. This relation may be written in the form

(4) $B^q_{mn} = q^2 (A^q_{m-1,n-1} - A^q_{m,n-1} - A^q_{m-1,n} + A^q_{mn})$ $- q(A^q_{m,n-1} + A^q_{m-1,n} - 2A^q_{mn}) + A^q_{mn}$.

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From this relation (4) it follows that if \( \lim_{m,n \to \infty} A_{mn}^q = A \), then \( \lim_{m,n \to \infty} B_{mn}^q = A \).

Relation (3) can be used to express \( A_{mn}^q \) in terms of \( B_{mn}^q \). First write (3) in the form

\[
(q + 1)^2 A_{mn}^q = B_{mn}^q + q(q + 1)(A_{m,n-1}^q + A_{m-1,n}^q) - q A_{m-1,n-1}^q.
\]

In this replace \( A_{m,n-1}^q \) and \( A_{m-1,n}^q \) by the values which this relation gives for them. This yields

\[
(q + 1)^2 A_{mn}^q = B_{mn}^q + q(q + 1)^{-1}(B_{m,n-1}^q + B_{m-1,n}^q)
\]

\[
+ q \left( A_{m,n-2}^q + A_{m-1,n-1}^q + A_{m-2,n}^q \right)
\]

\[
- q(q + 1)^{-1}(A_{m-1,n-2}^q + A_{m-2,n-1}^q).
\]

Successive repetitions of this procedure lead finally to the relation

\[
(q + 1)^2 A_{mn}^q = \sum_{h,k=0}^{m,n} \left( \frac{q}{q+1} \right)^{m+n-h-k} B_{hk}^q.
\]

Relation (5) expresses \( A_{mn}^q \) as a transform of the sequence \( B_{mn}^q \). The coefficients of the transformation satisfy the conditions for regularity [3, p. 23]. Hence if \( B_{mn}^q \) has the limit \( A \) and is bounded, then \( A_{mn}^q \) also has the limit \( A \).

To see that there exist sequences for which the transform \( B_{mn}^q \) has a limit but for which \( \lim_{m,n \to \infty} A_{mn}^q \) does not exist for any \( q \geq 0 \) consider the sequence \( A_{mn}^q = (-1)^n q^{m+n-1} \left( \frac{n(p+1)+p}{q+1} \right) \), \( p > 1 \). For this sequence one may readily verify by substitution into (2) that \( B_{mn}^q = 0 \) whenever \( n > 1 \). Thus for this sequence the transform \( B_{mn}^q \) has the limit 0. But by substituting into (1) and simplifying one obtains

\[
A_{mn}^q = \left\{ \left( \frac{q+p}{q+1} \right)^{m+1} - \left( \frac{q}{q+1} \right)^{m+1} \right\}
\]

\[
\cdot \left( \frac{n+1}{q+1} \right) \cdot \left( \frac{p+1}{p^2} \right) \cdot \left( \frac{q-p}{q+1} \right)^n
\]

\[
+ p^{-2} \left\{ \left( \frac{q+p}{q+1} \right)^{m+1} - \left( \frac{q}{q+1} \right)^{m+1} \right\}
\]

\[
\cdot \left( \frac{q-p}{q+1} \right)^n + \left( \frac{q}{q+1} \right)^{m+1}
\]

and for \( p > 1 \) this does not have a limit for any \( q \geq 0 \). This completes the proof of Theorem 1.
Theorem 2. Let $A_{m,n} = \sum_{h,k=0}^{m,n} a_{hk}$. If

$$B_{m,n}^1 = 2^{m-n} \sum_{h,k=0}^{m,n} \left( \begin{array}{c} m \\ h \end{array} \right) \left( \begin{array}{c} n \\ k \end{array} \right) A_{hk}$$

has the limit $A$ as $m, n \to \infty$, (2) $A_{m,n}$ is bounded and

$$\lim_{m,n \to \infty} (m^{1/2} + n^{1/2})(mn)^{1/2} a_{mn} = 0,$$

then $A_{m,n}$ also has the limit $A$ as $m, n \to \infty$.

Form the difference

$$B_{4m,4n}^1 - A_{2m,2n} = 2^{-4m-4n} \sum_{h,k=0}^{4m,4n} \left( \begin{array}{c} 4m \\ h \end{array} \right) \left( \begin{array}{c} 4n \\ k \end{array} \right) (A_{hk} - A_{2m,2n}).$$

Separate this difference into 9 parts $S_1, S_2, \ldots, S_9$ corresponding respectively to the intervals of summation

$$\begin{align*}
S_1 &= \left( 0 \leq h \leq m, 0 \leq k \leq n \right), \\
S_2 &= \left( 0 \leq h \leq m, n < k < 3n \right), \\
S_3 &= \left( 0 \leq h \leq m, 3n \leq k \leq 4n \right), \\
S_4 &= \left( m < h < 3m, 0 \leq k \leq n \right), \\
S_5 &= \left( m < h < 3m, n < k < 3n \right), \\
S_6 &= \left( m < h < 3m, 3n \leq k \leq 4n \right), \\
S_7 &= \left( 3m \leq h \leq 4m, 0 \leq k \leq n \right), \\
S_8 &= \left( 3m \leq h \leq 4m, n < k < 3n \right), \\
S_9 &= \left( 3m \leq h \leq 4m, 3n \leq k \leq 4n \right).
\end{align*}$$

Since

$$2^{-4m} \sum_{h=0}^{4m} \left( \begin{array}{c} 4m \\ h \end{array} \right) = 1,$$

$$\lim_{m \to \infty} 2^{-4m} \sum_{h=0}^{m} \left( \begin{array}{c} 4m \\ h \end{array} \right) = 0 \quad \text{[2, p. 511]}, \\
\lim_{m \to \infty} 2^{-4m} \sum_{h=0}^{4m} \left( \begin{array}{c} 4m \\ h \end{array} \right) = 0,$$

and $A_{mn}$ is bounded it follows that each of the parts $S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9$ has the limit zero as $m, n \to \infty$. Thus if $S_6$ has the limit zero it will follow that the difference $B_{4m,4n}^1 - A_{2m,2n}$ has the limit zero.

Let $Q_{m,n}$ denote the largest of the numbers $((m+h)^{1/2} + (n+k)^{1/2}) \cdot ((m+h)(n+k))^{1/2} \cdot |a_{m+h,n+k}|$ for $m < h < 3m$ and $n < k < 3n$. Then for all $h, k$ in these intervals

$$|A_{hk} - A_{2m,2n}| \leq (|2m - h| \cdot 3n + |2n - k| \cdot 2m) \frac{Q_{mn}}{(m^{1/2} + n^{1/2})(mn)^{1/2}}.$$
if $mn \neq 0$. Hence

$$|S_b| \leq 2^{-4m-4n} \sum_{h, k=m+1, n+1} \binom{4m}{h} \binom{4n}{k} |2m - h| \cdot 3n$$

$$+ 2n \sum_{h=m+1}^{2n-1} |2m - h| \cdot \binom{4m}{h}$$

$$+ 2m \cdot 2^{-4n} \sum_{h=n+1}^{2m-1} |2n - k| \cdot \binom{4n}{k}$$

$$\leq \frac{Q_{mn}}{(m^{1/2} + n^{1/2}) \cdot (mn)^{1/2}}.$$ 

But

$$\sum_{h=m+1}^{2n-1} |2m - h| \cdot \binom{4m}{h} < 2 \sum_{h=0}^{2m} (2m - h) \binom{4m}{h}$$

and

$$\sum_{h=0}^{2m} (2m - h) \binom{4m}{h}$$

$$= 2m \left\{ \frac{1}{2} \sum_{h=0}^{4m} \binom{4m}{h} + \frac{1}{2} \binom{4m}{2m} \right\} - 4m \sum_{h=1}^{2m-1} \binom{4m - 1}{h} - 1$$

$$= m \left\{ 2^{4m} + \binom{4m}{2m} - 4 \sum_{h=0}^{2m-1} \binom{4m - 1}{h} \right\} = m \binom{4m}{2m}.$$ 

Hence

$$|S_b| < \left\{ 6mn \cdot 2^{-4m} \binom{4m}{2m} + 4mn \cdot 2^{-4n} \binom{4n}{2n} \right\} \frac{Q_{mn}}{(m^{1/2} + n^{1/2}) \cdot (mn)^{1/2}}.$$ 

Since

$$2^{-2n} \binom{2n}{n} \approx (\pi n)^{-1/3}$$

[2, p. 385] it then follows that

$$|S_b| < \left\{ 6mn (2\pi m)^{-1/3} (1 + e_m) \right\} \frac{Q_{mn}}{(m^{1/2} + n^{1/2}) \cdot (mn)^{1/2}} + 4mn (2\pi n)^{-1/3} (1 + e_n) \frac{Q_{mn}}{(m^{1/2} + n^{1/2}) \cdot (mn)^{1/2}}$$

where $e_m \to 0$ as $m \to \infty$ and $e_n \to 0$ as $n \to \infty$. Thus
\[ |S_\delta| < \left\{ \frac{6n^{1/2}(1 + e_m) + 4m^{1/2}(1 + e_n)}{m^{1/2} + n^{1/2}} \right\} \cdot Q_{mn}. \]

Since the quantity in braces is bounded and \( Q_{mn} \to 0 \) it then follows that \( S_\delta \to 0 \) as \( m, n \to \infty \). Hence the difference \( B_{1m,2n}^1 - A_{2m,2n} \) has the limit zero. With only slight modifications of this argument it can be shown that \( B_{1m+1,2n}^1 - A_{2m+1,2n} \), \( B_{1m,2n+1}^1 - A_{2m,2n+1} \), and \( B_{1m+1,2n+1}^1 - A_{2m+1,2n+1} \) have the limit zero. The proof of the theorem is then complete.

References


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