ON EULER METHODS OF SUMMABILITY FOR DOUBLE SERIES

G. N. WOLLAN

The two \( q \)th order Euler transforms of the sequence \( A_n \)

\[
A_n^q = (q + 1)^{\frac{n}{q}} \sum_{k=0}^{n} \binom{n+1}{k+1} q^{n-k} A_k
\]

and

\[
B_n^q = (q + 1)^{\frac{n}{q}} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} A_k
\]

are equivalent for \( q \geq 0 \) in the sense that if either has a limit as \( n \to \infty \) the other has the same limit \([1, p. 180]\).\(^1\) For double sequences the corresponding transforms are

\[
(1) \quad A_{mn}^q = (q + 1)^{\frac{m-n-2}{q}} \sum_{h,k=0}^{m,n} \binom{m+1}{h+1} \binom{n+1}{k+1} q^{m+n-h-k} A_{hk},
\]

\[
(2) \quad B_{mn}^q = (q + 1)^{\frac{m-n}{q}} \sum_{h,k=0}^{m,n} \binom{m}{h} \binom{n}{k} q^{m+n-h-k} A_{hk}.
\]

This paper is concerned with two theorems regarding these transforms. Throughout the discussion \( q \geq 0 \).

**Theorem 1.** If \( A_{mn}^q \) has a limit as \( m, n \to \infty \), then \( B_{mn}^q \) has that same limit and if \( B_{mn}^q \) has a limit and is bounded, then \( A_{mn}^q \) has that same limit but there do exist sequences for which \( B_{mn}^q \) has a limit but for which \( \lim_{m,n \to \infty} A_{mn}^q \) does not exist for any \( q \geq 0 \).

The relation

\[
B_{mn}^q = q^2 A_{m-1,n-1}^q - q(q + 1)(A_{m,n-1}^q + A_{m-1,n}^q) + (q + 1)^2 A_{mn}^q
\]

may be verified by substitution from (1) into the right-hand side. This relation may be written in the form

\[
B_{mn}^q = q^2 (A_{m-1,n-1}^q - A_{m,n-1}^q - A_{m-1,n}^q + A_{mn}^q) - q(A_{m,n-1}^q + A_{m-1,n}^q - 2A_{mn}^q) + A_{mn}^q.
\]

Presented to the Society, November 29, 1952; received by the editors December 4, 1952.

\(^1\) Numbers in brackets refer to the references at the end of the paper.

583
From this relation (4) it follows that if \( \lim_{m,n \to \infty} A_{m,n}^q = A \), then \( \lim_{m,n \to \infty} B_{m,n}^q = A \).

Relation (3) can be used to express \( A_{m,n}^q \) in terms of \( B_{m,n}^q \). First write (3) in the form

\[
(q + 1) A_{m,n}^q = B_{m,n}^q + q(q + 1)(A_{m,n-1}^q + A_{m-1,n}^q) - q A_{m-1,n-1}^q.
\]

In this replace \( A_{m,n-1}^q \) and \( A_{m-1,n}^q \) by the values which this relation gives for them. This yields

\[
(q + 1) A_{m,n}^q = B_{m,n}^q + q(q + 1)(A_{m,n-2}^q + A_{m-1,n-1}^q + A_{m-2,n}^q) - q(q + 1)(A_{m-1,n-2}^q + A_{m-2,n-1}^q).
\]

Successive repetitions of this procedure lead finally to the relation

\[
(q + 1)^2 A_{m,n}^q = \sum_{k=0}^{m,n} \left( \frac{q}{q + 1} \right)^{m+n-k} B_{h,k}^q.
\]

Relation (5) expresses \( A_{m,n}^q \) as a transform of the sequence \( B_{m,n}^q \). The coefficients of the transformation satisfy the conditions for regularity [3, p. 23]. Hence if \( B_{m,n}^q \) has the limit \( A \) and is bounded, then \( A_{m,n}^q \) also has the limit \( A \).

To see that there exist sequences for which the transform \( B_{m,n}^q \) has a limit but for which \( \lim_{m,n \to \infty} A_{m,n}^q \) does not exist for any \( q \geq 0 \) consider the sequence \( A_{m,n} = (-1)^n p^{m+n-1} \{ n(p+1) + p \} \), \( p > 1 \). For this sequence one may readily verify by substitution into (2) that \( B_{m,n}^q = 0 \) whenever \( n > 1 \). Thus for this sequence the transform \( B_{m,n}^q \) has the limit 0. But by substituting into (1) and simplifying one obtains

\[
A_{m,n}^q = \left\{ (q + p)^{m+1} \left( \frac{q}{q + 1} \right)^{m+1} \right\} \left( \frac{n + 1}{q + 1} \right) \left( \frac{p + 1}{p^2} \right) \left( \frac{q - p}{q + 1} \right)^n
\]

\[
+ p^{-k} \left\{ (q + p)^{m+1} \left( \frac{q}{q + 1} \right)^{m+1} \right\} \left( \frac{q - p}{q + 1} \right)^n
\]

and for \( p > 1 \) this does not have a limit for any \( q \geq 0 \). This completes the proof of Theorem 1.
Theorem 2. Let $A_{mn} = \sum_{h,k=0}^{m,n} a_{hk}$. If

$$(1) \quad B_{mn}^1 = 2^{-m-n} \sum_{h,k=0}^{m,n} \binom{m}{h} \binom{n}{k} A_{hk}$$

has the limit $A$ as $m, n \to \infty$, (2) $A_{mn}$ is bounded and

$$(3) \quad \lim_{m,n \to \infty} (m^{1/2} + n^{1/2})(mn)^{1/2} a_{mn} = 0,$$

then $A_{mn}$ also has the limit $A$ as $m, n \to \infty$.

Form the difference

$$B_{4m,4n}^1 - A_{2m,2n} = 2^{-4m-4n} \sum_{h,k=0}^{4m,4n} \binom{4m}{h} \binom{4n}{k} (A_{hk} - A_{2m,2n}).$$

Separate this difference into 9 parts $S_1, S_2, \ldots, S_9$ corresponding respectively to the intervals of summation

$$(0 \leq h \leq m), \quad (0 \leq k \leq n), \quad (0 \leq h \leq m), \quad (3n \leq k \leq 4n),$$

$$(m < h < 3m), \quad (m < k < 3n), \quad (m < h < 3m), \quad (3n \leq k \leq 4n),$$

$$(3m \leq h \leq 4m), \quad (3m \leq k \leq 4n), \quad (3m \leq h \leq 4m), \quad (3n \leq k \leq 4n).$$

Since

$$2^{-4m} \sum_{h=0}^{4m} \binom{4m}{h} = 1,$$

$$\lim_{m \to \infty} 2^{-4m} \sum_{h=0}^{m} \binom{4m}{h} = 0 \quad [2, p. 511], \quad \lim_{m \to \infty} 2^{-4m} \sum_{h=0}^{4m} \binom{4m}{h} = 0,$$

and $A_{mn}$ is bounded, it follows that each of the parts $S_1, S_2, S_3, S_4,$

$S_5, S_6, S_7, S_8, S_9$ has the limit zero as $m, n \to \infty$. Thus if $S_6$ has the limit zero it will follow that the difference $B_{4m,4n}^1 - A_{2m,2n}$ has the limit zero.

Let $Q_{m,n}$ denote the largest of the numbers $((m+h)^{1/2} + (n+k)^{1/2})

\cdot ((m+h)(n+k))^{1/2} \cdot |a_{m+h,n+k}|$ for $m < h < 3m$ and $n < k < 3n$. Then for all $h, k$ in these intervals

$$|A_{hk} - A_{2m,2n}|$$

$$\leq \left( |2m - h| \cdot 3n + |2n - k| \cdot 2m \right) \frac{Q_{mn}}{(m^{1/2} + n^{1/2})(mn)^{1/2}}$$
if $mn \neq 0$. Hence

$$|S_b| \leq 2^{-4m-4n} \sum_{k, n=n+1, n+1}^{m-1, m-1} \binom{4m}{h} \binom{4n}{k} |2m - h| \cdot 3n$$

$$+ |2n - k| \cdot 2m) \frac{Q_{mn}}{(m^{1/2} + n^{1/2} \cdot (mn)^{1/2}}$$

$$\leq \left\{ 3n \cdot 2^{-4m} \sum_{h=n+1}^{m-1} |2m - h| \cdot \binom{4n}{h} \right. \right.$$  

$$+ \left. 2m \cdot 2^{-4m} \sum_{h=n+1}^{m-1} |2n - k| \cdot \binom{4n}{h} \right\} \frac{Q_{mn}}{(m^{1/2} + n^{1/2} \cdot (mn)^{1/2}}.$$

But

$$\sum_{h=n+1}^{m-1} |2m - h| \cdot \binom{4m}{h} < 2 \sum_{h=0}^{2m} (2m - k) \binom{4m}{h}$$

and

$$\sum_{h=0}^{2m} (2m - k) \binom{4m}{h}$$

$$= 2m \left\{ \frac{1}{2} \sum_{h=0}^{4m} \binom{4m}{h} + \frac{1}{2} \binom{4m}{2m} - 4m \sum_{h=0}^{2m} \binom{4m - 1}{h - 1} \right\}$$

$$= m \left\{ 2^{4m} + \binom{4m}{2m} - 4 \sum_{h=0}^{2m} \binom{4m - 1}{h} \right\} = m \binom{4m}{2m}.$$

Hence

$$|S_b| < \left\{ 6mn \cdot 2^{-4m} \binom{4m}{2m} + 4mn \cdot 2^{-4m} \binom{4n}{2n} \right\} \frac{Q_{mn}}{(m^{1/2} + n^{1/2} \cdot (mn)^{1/2}}.$$

Since

$$2^{-2m} \binom{2m}{n} \leq (\pi n)^{-1/3}$$

[2, p. 385] it then follows that

$$|S_b| < \left\{ 6mn(2\pi m)^{-1/3}(1 + e_m)$$

$$+ 4mn(2\pi n)^{-1/3}(1 + e_n) \right\} \frac{Q_{mn}}{(m^{1/2} + n^{1/2} \cdot (mn)^{1/2}}$$

where $e_m \to 0$ as $m \to \infty$ and $e_n \to 0$ as $n \to \infty$. Thus
\[ |S_6| < \left\{ \frac{6n^{1/2}(1 + e_n) + 4m^{1/2}(1 + e_n)}{m^{1/2} + n^{1/2}} \right\} \cdot Q_{mn}. \]

Since the quantity in braces is bounded and \( Q_{mn} \to 0 \) it then follows that \( S_6 \to 0 \) as \( m, n \to \infty \). Hence the difference \( B_{m,n}^1 - A_{2m,2n} \) has the limit zero. With only slight modifications of this argument it can be shown that \( B_{m,n}^1 - A_{2m+1,2n}, B_{m,n}^1 - A_{2m,2n+1} \), and \( B_{m,n}^1 - A_{2m+1,2n+1} \) have the limit zero. The proof of the theorem is then complete.

References

Memphis State College