A NONCONVERGENT ITERATIVE PROCESS

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1. Introduction. In [1] Mann and Wolf considered the integral equation

$$y(t) = \int_0^t \frac{G[y(x)]}{\pi(t - x)^{1/2}} \, dx,$$

where

$$G(y)$$ is continuous and strictly decreasing for positive $$y$$, and

$$G(1) = 0.$$  

They defined a sequence of functions $$y_0(t), y_1(t), \ldots$$ inductively as follows:

$$y_0(t) = 0, \quad y_{n+1}(t) = \int_0^t \frac{G[y_n^*(x)]}{\pi(t - x)^{1/2}} \, dx,$$

where $$y_n^*(x) = \min(y_n(x), 1)$$. Under the additional assumption that $$G(y)$$ satisfies a Lipschitz condition on $$[0, 1]$$ they proved that the sequence $$y_0(t), y_1(t), \ldots$$ converges to a bounded solution,\(^1\) $$y(t)$$, of (1). Dr. Mann pointed out to me that it was not known whether or not the requirement of a Lipschitz condition was superfluous. The present paper resolves this uncertainty by giving an example of a function $$G(y)$$ satisfying (2) for which the corresponding sequence (3) does not converge. It also contains a positive result, to the effect that the sequence defined by (3) does converge to the solution $$y(t)$$ if, in addition to requirement (2), $$G(y)$$ is convex.

2. The counter example. The desired function $$G(y)$$ is defined as follows:

$$G(y) = 1 - y \quad \text{for } 0 \leq y \leq 1/2;$$

$$G(y) = \frac{1 - (2y - 1)^{1/2}}{2} \quad \text{for } 1/2 < y.$$  

Let $$G_1(y) = 1 - y$$ for $$y \geq 0$$, and let $$z(t)$$ be the bounded solution of

\(^1\) It was shown in [2] that even in the absence of the Lipschitz condition equation (1) has a unique bounded solution $$y(t)$$, provided only that $$G$$ satisfies requirement (2). This solution $$y(t)$$ is strictly increasing and approaches the limit 1 as $$t$$ increases indefinitely.
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(5) \[ z(t) = \int_0^t \frac{G[z(x)]}{[\pi(t - x)]^{1/3}} \, dx. \]

Now, as was shown in [1, p. 168], \( z(t) \) is continuous \( t \geq 0 \) and \( ds/dt \) is positive and decreasing for \( t > 0 \). Thus a positive number \( \alpha \) is uniquely determined by the requirement \( z(\alpha) = 1/2 \). Let \( k \) be \( ds/dt \) evaluated at \( 2\alpha \), and let \( \beta \) be the smaller of \( 2\alpha \) and \( \alpha + 4\alpha^2 k^2 \). Then clearly

(6) \[ z(t) \geq 1/2 + k(t - \alpha) \quad (\alpha \leq t \leq \beta). \]

Lemmas 1 and 2. If, for some \( n \), \( y_n(t) \geq z(t) \) for \( 0 \leq t \leq \beta \), then over the same interval \( y_{n+1}(t) \leq \min (z(t), 1/2) \).

Lemmas 3 and 4. If, for some \( n \), \( y_n(t) \leq \min (z(t), 1/2) \) for \( 0 \leq t \leq \beta \), then over the same interval \( y_{n+1}(t) \geq z(t) \).

Assuming that these lemmas are true, then \( y_n(t) \leq \min (z(t), 1/2) \), and \( y_{n+1}(t) \geq z(t) \), on \( 0 \leq t \leq \beta \). Then clearly the sequence \( y_0(t), y_1(t), \ldots \) does not converge for any \( t \) between \( \alpha \) and \( \beta \) because \( z(t) > 1/2 \) over this range.

Proof of Lemma 1. Define \( Y(t) \) as follows:

(7) \[ Y(t) = \int_0^t \frac{G[z(x)]}{[\pi(t - x)]^{1/3}} \, dx. \]

Since \( y_n(x) \geq z(x) \) for \( 0 \leq x \leq \beta \) and \( G \) is a decreasing function, we have \( y_{n+1}(t) \leq Y(t) \). Now \( Y(t) = z(t) \) for \( 0 \leq t \leq \alpha \); we shall show that \( Y(t) < 1/2 \) for \( \alpha < t \leq \beta \). Throughout the remainder of the proof \( t \) will be a fixed number \( \alpha + \Delta t, 0 < \Delta t \leq \beta - \alpha \). From (7) we have

(8) \[ Y(t) - Y(\alpha) = \int_\alpha^{\alpha + \Delta t} \frac{G[z(x)]}{[\pi(\alpha + \Delta t - x)]^{1/3}} \, dx \]

\[ - \int_\alpha^\alpha \frac{G[z(x)]}{[\pi(\alpha - x)]^{1/3}} \left[ \frac{1}{[\pi(\alpha + \Delta t - x)]^{1/3}} \right] \, dx, \]

\[ = \text{Gain} - \text{Loss}, \text{say}. \]

Now \( G[z(x)] \geq 1/2 \) over \( 0 \leq x \leq \alpha \), and integration gives

(9) \[ \text{Loss} \geq \pi^{-1/2}(\alpha^{1/2} - (\alpha + \Delta t)^{1/2} + (\Delta t)^{1/2}). \]

To get an upper bound on the gain in (8) we first use (6) and (4),
and find that $G[z(x)] \leq (1 - [2k(x - \alpha)]^{1/2})/2 = f(x)$, say. Next we replace $f(x)$ by the linear function $F(x)$ determined to equal $f(x)$ at $x = \alpha$ and at $x = \alpha + \Delta t$. Since $f(x)$ is convex we clearly have $f(x) \leq F(x)$ ($\alpha \leq x \leq \alpha + \Delta t$), and thus

$$G[z(x)] \leq F(x) = 1/2 - m(x - \alpha),$$

where $m = (\Delta t)^{-2/3}(2k)^{1/2}/2$. Substituting for $G[z(x)]$ in the first integral of (8) and performing the integration gives

$$\text{Gain} \leq \pi^{1/2}[(\Delta t)^{1/2} - (4m/3)(\Delta t)^{3/2}].$$

Then, from (11) and (9),

$$\text{Gain} - \text{Loss} \leq \pi^{-1/2}[(\alpha + \Delta t)^{1/2} - \alpha^{1/2} - (4m/3)(\Delta t)^{3/2}]$$
$$\leq \pi^{-1/2}[\Delta t(\alpha^{1/2}/2) - (4m/3)(\Delta t)^{3/2}]$$
$$= (\pi^{-1/2}\Delta t)[(\alpha^{1/2}/2) - (4m/3)(\Delta t)]^{3/2}.$$  

In the last member of (12) replace $m$ by its value (see (10)), and replace $\Delta t$ (in the second factor) by its upper bound, $4\alpha^2k^2$. This gives

$$\text{Gain} - \text{Loss} \leq \pi^{-1/2}\Delta t[\alpha^{1/2}/2 - 2\alpha^{-1/2}/3] < 0.$$  

This completes the proof of Lemma 1.

**Proof of Lemma 2.** Now $G(y) = G_1(y)$ for $0 \leq y \leq 1/2$, so under the hypothesis that $y_n(t) \leq \min (z(t), 1/2)$ we know that

$$G[y_n(x)] = G_1[y_n(x)]$$

for $0 \leq x \leq \beta$.

Then over this range

$$y_{n+1}(t) = \int_0^t G_1[y_n(x)] \frac{dx}{\pi(t - x)^{1/2}} \geq \int_0^t G_1[z(x)] \frac{dx}{\pi(t - x)^{1/2}} = z(t).$$

With this proof of Lemma 2 our discussion of the counterexample is complete.

3. **The theorem.** If $G(y)$ satisfies (2) and in addition is convex for $0 \leq y \leq 1$, then the sequence $y_0(t), y_1(t), \cdots$ given by (3) converges to the solution $y(t)$ of (1).

**Proof.** Now $y_0(t) = 0$ and $y_1(t) = 2G(0)(t/\pi)^{1/2}$. Define positive numbers $d$ and $c$ by the respective requirements

$$G(d) = 3G(0)/4, \quad y_1(c) = d.$$  

We first prove the conclusion of the theorem for $t$ restricted to the interval $[0, c]$. 

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From the convexity of \( G(y) \) we see that for any \( r_1 \) and \( r_2 \) between 0 and 1 (\( r_1 \neq r_2 \)) we have

\[
(15) \quad \frac{G(r_1) - G(r_2)}{r_1 - r_2} \leq \frac{G(0) - G(r_1)}{r_1}.
\]

From our choice of \( d \) and \( c \) and the fact that \( y_n(t) \leq y_1(t) \) for all \( n \) we see from (2) that \( G[y_n(t)] \geq 3G(0)/4 \) for \( 0 \leq t \leq c \). Then

\[
(16) \quad y_n(t) \geq (3/4)y_1(t) = (3/2)G(0)(t/\pi)^{1/2}.
\]

Let \( \Delta_n = \max |y_{n+1}(t) - y_{n-1}(t)| \) for \( 0 \leq t \leq c \). Thus

\[
|y_{n+1}(t) - y_n(t)| = \int_0^t \frac{|G[y_n(x)] - G[y_{n-1}(x)]|}{[\pi(t - x)]^{1/2}} \, dx
\]

\[
\leq \int_0^t \frac{G(0)/4}{y_n(x)[\pi(t - x)]^{1/2}} \, dx
\]

\[
= \frac{\Delta_n}{6} \int_0^t \frac{\pi}{[\pi(t - x)]^{1/2}} \, dx = (\pi/6)\Delta_n.
\]

(In the above we first use (3), then (15), and then (14) and (16). The two final equalities are obvious.) Thus \( |y_{n+1}(t) - y_n(t)| \leq \Delta_{n+1} \leq \Delta_n(\pi/6) \leq (\pi/6)^n \), for \( 0 \leq t \leq c \). This proves the convergence on the interval \([0, c]\).

Suppose the theorem is false and that on some interval \([0, T]\) the sequence \( y_0(t), y_1(t), \ldots \) does not converge. Now for every \( t \), \( y_0(t) \leq y_2(t) \leq \ldots \leq y(t) \leq \ldots \leq y_1(t) \leq y_1(t) \). Therefore the \( y \)'s of even subscript converge to a continuous limit function \( Y_1(t) \) and the \( y \)'s of odd subscript converge to a continuous limit function \( Y_2(t) \), and \( Y_1(t) \leq y(t) \leq Y_2(t) \). It is furthermore clear that the substitutition of \( Y_1(x) \) [respectively \( Y_2(x) \)] for \( y(x) \) under the integral sign in (1) gives \( Y_2(t) \) [respectively \( Y_1(t) \)] in place of \( y(t) \). The convergence of \( y_0(t) \), \( y_1(t), \ldots \) on \([0, c]\) implies that \( Y_1(t) = Y_2(t) \) for \( 0 \leq t \leq c \). Let \( e \) be the greatest number such that \( Y_1(t) = Y_2(t) \) for \( 0 \leq t \leq e \). Then \( c \leq e < T \).

Since \( Y_1 \) has a positive minimum value on \([e, T]\) it follows from the hypotheses on \( G \) that there exists a positive \( k \) such that for any \( x \) on \([e, T]\), \( G[Y_1(x)] - G[Y_2(x)] \leq k |Y_1(x) - Y_2(x)| \). Choose a fixed \( t \) \((e < t < T)\) so that (a) \( 2k[(t-e)/\pi]^{1/2} < 1 \) and (b) \( |Y_1(x) - Y_2(x)| \leq |Y_1(t) - Y_2(t)| \) for \( e \leq x \leq t \). Then
\[ |Y_2(t) - Y_1(t)| = \int_0^t \frac{G[Y_1(x)] - G[Y_2(x)]}{[\pi(t - x)]^{1/2}} \, dx \]
\[ \leq \int_0^t k \frac{|Y_1(x) - Y_2(x)|}{[\pi(t - x)]^{1/2}} \, dx \]
\[ \leq k |Y_1(t) - Y_2(t)| \int_0^t \frac{dx}{[\pi(t - x)]^{1/2}} \]
\[ = 2k[(t - \varepsilon)/\pi]^{1/2} \cdot |Y_1(t) - Y_2(t)| \]
\[ < Y_2(t) - Y_1(t). \]

Thus the assumption that the theorem is false has led to a contradiction.

**REFERENCES**


**Duke University**