RATIONAL NORMAL MATRICES SATISFYING THE INCIDENCE EQUATION

A. A. ALBERT

1. Introduction. An incidence matrix $A$ of a finite projective plane of order $m$ is an $n$-rowed square matrix $A$ with nonnegative integral elements such that

$$B = AA' = mI + N,$$

where $n = m^2 + m + 1$, $I$ is the $n$-rowed identity matrix, and all elements of $N$ are 1. It can then be shown that every element of $A$ is either 0 or 1, that there are precisely $m+1$ nonzero elements in every row and column of $A$, and that it follows that

$$A'A = B.$$

Thus an incidence matrix is a normal integral matrix satisfying the incidence equation (1).

The following result is also known:

BRUCK-RYSER THEOREM. Let $m = \not\equiv 1, 2 \pmod{4}$, and let there exist a rational matrix $P$ satisfying the incidence equation $PP' = mI + N$. Then $m$ is a sum of two squares.

The converse of this theorem is also true and provides what may be thought of as a rational approximation to an incidence matrix. The purpose of this note is that of giving a constructive proof of the following closer approximation.

Theorem. Let $m$ be a sum of two squares. Then there exists a normal matrix $S$ with rational elements such that $SS' = mI + N$.

2. Algebraic properties. If $PP' = SS' = B$, then $(P^{-1}S)(P^{-1}S)' = I'$. Hence, if $P$ and $S$ are any two solutions of the incidence equation, there exists an orthogonal matrix $C$ such that

$$S = PC.$$

When $P$ and $S$ are rational solutions the orthogonal matrix $C$ must also be rational. Conversely if $S = PC$, where $C$ is orthogonal and $P$ satisfies the incidence equation, then $S$ satisfies the incidence equation. We note the following stronger result:

Received by the editors November 17, 1952.

Lemma 1. The matrix $S = PC$ is normal if and only if $C'P'PC = PP'$. When $S$ is a normal solution of the incidence equation the matrix $T = SG$ is also a normal solution if and only if $G$ is an orthogonal matrix such that the sum of the elements in every row and column of either $G$ or $-G$ is 1.

For if $S$ is normal we see that $SS' = PP' = S'S = C'(P'P)C$. If $T = SG$ is a second normal solution, then $T'T = G'S'SG = TT' = G'(SS')G$, that is, $G'BG = B$. But $B = mI + N$, and the orthogonal matrix $G$ commutes with $B$ if and only if

$$GNG' = N, \quad GN = NG.$$ 

However

$$N = u'u, \quad u = (1, 1, \ldots, 1),$$

and (4) is equivalent to

$$N = v'v, \quad v = uG.$$ 

The $i$th element of the row vector $v$ is the sum $s_i$ of the elements in the $i$th column of $G$, and (6) implies that $s_is_j = 1$. Hence $s_i^2 = 1$ and $s_i = 1$ or $-1$. Since $s_is_j = 1$ the sums $s_i$ have the same sign and are equal. The second form of (4) implies that the sum of the elements in the $i$th row of $G$ is equal to the column sum $s_i$, and our result is proved.

3. A rational solution and a basic equation. We shall assume henceforth that

$$m = a^2 + b^2,$$

for integers $a$ and $b$. Then the $n$-rowed square matrix

$$P = \begin{bmatrix} 0 & c & c & \cdots & c \\ d' & H & 0 & \cdots & 0 \\ d' & 0 & H & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ d' & 0 & 0 & \cdots & H \end{bmatrix}$$

defined by the formulas

$$c = \left(\frac{a-b}{m}, \frac{a+b}{m}\right), \quad d = (1, 1), \quad H = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

is a solution of the incidence equation. Indeed the length of the first row of $P$ is $kcc' = km^{-2}[((a-b)^2 + (a+b)^2)] = 2km^{-2}m = m+1$, where we
have introduced the notation

\[ k = \frac{m^2 + m}{2}. \]

The length of every other row is \(1 + a^2 + b^2 = 1 + m\) and so the diagonal elements of \(PP'\) are \(m + 1\). The inner product of the \(i\)th row of \(P\) and the \(j\)th row is \(1\) trivially for \(i > j > 1\). The remaining inner products are \([a(a - b) + b(a + b)]m^{-1} = (a^2 + b^2)m^{-1} = 1\) and \([-b(a - b) + a(a + b)]m^{-1} = 1\), and so we have proved that

\[ PP' = B. \]

Let us now compute

\[ P'P = mI + M. \]

By direct computation using (8) we see that

\[ M = \frac{1}{m^2} w'w, \]

where

\[ w = (m^2, a - b, a + b, \ldots, a - b, a + b). \]

Observe that \(ww' = m^4 + k [(a - b)^2 + (a + b)^2] = m^4 + m(m^2 - m)\), that is,

\[ ww' = m^2n. \]

We shall attempt to find a rational orthogonal matrix \(C\) such that \(PC\) is a normal matrix. Our success will depend on a rational solution of the equation \(x^2 - my^2 = -n\), and we shall write the result as

\[ t^2 - ms^2 = -na^2, \]

for integers \(s\) and \(t\). To compute \(s\) and \(t\) we note that \((m + 1)^2 - m(1)^2 = m^2 + 2m + 1 - m = n\), and that \(b^2 - m(1)^2 = -a^2\). But then \((m + 1 + m^{1/2})(b + m^{1/2}) = t + sm^{1/2}\) where

\[ t = b(m + 1) + m, \quad s = b + (m + 1). \]

It should now be clear that \(t^2 - ms^2 = -na^2\).

4. A rational normal solution. We shall determine \(C\) as the product \(C' C_0\), where \(C_0\) and \(C_1\) are orthogonal matrices such that

\[ C_0 NC_0' = C_1 MC_1' = \begin{pmatrix} 0 & 0 \\ 0 & n \end{pmatrix}. \]
Moreover
\[(19)\quad C_0 = D_0^{-1}E_0, \quad C_1 = D_1^{-1}E_1,\]
where $E_0$ and $E_1$ will be taken to be \textit{integral} matrices, $D_0$ and $D_1$ will be taken to be \textit{diagonal} matrices. It will then follow that
\[(20)\quad C = E_1'(D_0D_1)^{-1}E_0\]
will be rational if and only if $D_0D_1$ is rational.

Write
\[(21)\quad \phi_1 = (0, 1, 0, -1, 0, \cdots, 0), \]
\[(21)\quad \phi_2 = (0, 1, 0, 1, 0, -2, \cdots, 0), \]
\[(21)\quad \phi_i = (0, 1, 0, 1, 0, -i, 0, \cdots, 0), \cdots, \]
\[(21)\quad \phi_k = (0, 1, 0, 1, \cdots, 0, 1, 0, 1 - k, 0). \]

Thus $\phi_i$ has $i$ elements 1, followed by the element $-i$, and these elements are separated by zeros. Since the rows of $N$ are all equal it should be clear that $\phi_iN = 0$. But it is actually evident that
\[(22)\quad \phi_iN = \phi_iM = 0. \]

Similarly we write
\[(23)\quad \eta_j = (0, 0, 1, 0, 1, \cdots, 0, 1, 0, -j, \cdots, 0) \quad (j = 1, \cdots, k - 1) \]
and have
\[(24)\quad \eta_jN = \eta_jM = 0. \]

Define
\[(25)\quad E_0 = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_k \\ q_1 \\ \vdots \\ q_k \end{bmatrix}, \quad E_1 = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_k \\ q_1 \\ \vdots \\ q_k \end{bmatrix}, \]
where we have already defined $k = (m^2 + m)/2$, $u = (1, 1, \cdots, 1)$, and $w = (m^2, a - b, a + b, \cdots, a - b, a + b)$. Define
\[ z = (0, a + b, b - a, a + b, b - a, \ldots, a + b, b - a) \]

and

\[ v = (-m - 1, a - b, a + b, a - b, a + b, \ldots, a - b, a + b). \]

The first \( n - 3 \) rows of \( E_0 \) coincide with those of \( E_1 \) and are clearly pairwise orthogonal characteristic vectors of both \( N \) and \( M \). The condition that a vector \( x = (x_1, \ldots, x_n) \) shall be orthogonal to \( p_1, \ldots, p_{k-1}, q_1, \ldots, q_{k-1} \) is that

\[ x_2 = x_4 = x_6 = \cdots = x_{n-1}, \quad x_3 = x_5 = \cdots = x_n, \]

and \( w, z \) and \( v \) satisfy this condition. By (13) we have

\[
\begin{align*}
zs &= \frac{1}{m^2} (zw')w = 0, \\
vM &= \frac{1}{m^2} vv'w = 0, \\
wM &= \frac{1}{m^2} w(w'w) = nw,
\end{align*}
\]

where it should be clear that \( zw' = k[(a+b)(a-b)+(b-a)(a+b)] \) = 0 = \( w' \) and that \( vv' = -m^2(m+1)+k(2m) = -m^2(m+1)+(m^2+m)m = 0. \]

It remains to compute the lengths of the rows of \( E_1 \). Clearly \( p_i p_i' \) = \( i^2 + i^2 = i(i+1) = q_i q_i' \). Next we see that \( zz' = k[(a+b)^2+(a-b)^2] = 2km = m^2(m+1) \) and that \( vv' = (m+1)^2+2km = (m+1)(m+1+m^2) = n(m+1) \). We have proved the following result:

**Lemma 2.** Let \( E_1 \) be given by (25) and \( D_1 \) be the diagonal matrix

\[
D_1 = \text{diag } \{ (1 \cdot 2)^{1/2}, (2 \cdot 3)^{1/2}, \ldots, ((k-1)k)^{1/2}, \ (1 \cdot 2)^{1/2}, \ (2 \cdot 3)^{1/2}, \ldots, ((k-1)k)^{1/2}, \ m(m+1)^{1/2}, \ (n(m+1))^1/2, \ mn^{1/2} \}.
\]

Then \( C_1 = D_1^{-1}E_1 \) is an orthogonal matrix such that \( C_1MC_1' \) satisfies (18).

We next write \( x = (x_1, \ldots, x_n) \) where

\[
\begin{align*}
x_1 &= -2ak, \quad x_2 = x_4 = \cdots = x_{n-1} = a + t, \\
x_3 &= x_5 = \cdots = x_n = a - t.
\end{align*}
\]

Then \( xx' = 4a^2k^2 + 2k(a^2 + t^2) = (m^2+m)[(m^2+m+1)a^2 + t^2] = (m^2+m)(na^2 + t^2) \). By (16) we have the value

\[
xx' = m^2s^2(m+1).
\]

We similarly write \( y = (y_1, \ldots, y_n) \), \( y_2 = y_4 = \cdots = y_{n-1} \), \( y_3 = y_5 = \cdots = y_n \) where

\[
\begin{align*}
y_1 &= -2kt, \quad y_2 = t - na, \quad y_3 = t + na.
\end{align*}
\]
Then \( yy' = 4k^2t^2 + k[(t-na)^2 + (t+na)^2] = (m^2+m)[(m^2+m)t^2 + t^2 + n^2a^2] = (m^2+m)(n^2+n^2a^2) \). Using (16) we have

\[
(34) \quad yy' = m^2s^2n(m + 1).
\]

The first \( n-3 \) rows of \( E_0 \) are already known to be pairwise orthogonal and orthogonal to \( x, y, u \). It should now be clear that since \( xu' = -2ka + k(a+t+a-t) = 0 \) and \( yu' = -2kt + k[t-na+t+na] = 0 \) the vectors \( x, y \) are orthogonal characteristic vectors of \( N = u'u \). Moreover

\[
xy' = \begin{cases}
   (-2k)^2at + k[(a+t)(t-na) + (a-t)(t+na)] & \\
   4k^2at + k(t^2 + at - na^2 - nat + at - t^2 + na^2 - nat)
\end{cases} = 4k^2at + 2kat(1-n) = 0
\]

since \( 1-n = -(m^2+m) = -2k \).

This completes our proof of the fact that the rows of the matrix \( E_0 \) form a set of \( n \) pairwise orthogonal characteristic vectors of \( N \). Define

\[
D_0 = \text{diag} \{ (1,2)^{1/2}, (2,3)^{1/2}, \ldots, ((k-1)k)^{1/2}, (1,2)^{1/2}, (2,3)^{1/2}, \ldots, ((k-1)k)^{1/2}, ms(m+1)^{1/2}, ms(n(m+1))^{1/2}, n^{1/2} \},
\]

and see that

\[
D = D_0D_1 = \text{diag} \{ 1, 2, 3, \ldots, k^2 - k, 1, 2, 3, \ldots, \}
\]

\[
k^2 - k, m^2s(m+1), ms(m+1), mn\}
\]

is an integral matrix. We have shown that for this \( D \) the matrix

\[
(37) \quad C = E_0D^{-1}E_0
\]

is a rational orthogonal matrix, and \( PC \) is a rational normal solution of the incidence equation. This completes our constructive proof.