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CONSTRUCTION OF SOLUTIONS AND PROPAGATION OF ERRORS IN NONLINEAR PROBLEMS

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1. **Introduction.** The purpose of this paper is to provide theoretical tools which will be useful, in practical applications, for approximating the solutions of a certain class of nonlinear equations.

2. **Definitions.** Let $(U, +, d)$ denote a mathematical system consisting of a set U on whose elements are defined (i) a binary operation, $+$, such that U is an Abelian group with respect to this operation and (ii) a metric function d, with respect to which U is a metric space. If d is invariant under translation (i.e., for any u_1, u_2, u_3, in U, $d(u_1, u_2)=d(u_1+u_3, u_2+u_3)$), then we shall call $(U, +, d)$ a *b-space.* Under these conditions we shall, for brevity, say that *U is a b-space.* The group identity will be denoted by θ, and $d(u, \theta)$ will be denoted by $\|u\|$.

If $\mathfrak{M}$ and $\mathfrak{N}$ are sets, and K is a single-valued function defined for each $\mathfrak{m}$ of $\mathfrak{M}$ and having its values (which are denoted by $K\mathfrak{m}$ or $K(\mathfrak{m})$) in $\mathfrak{N}$ then we shall say that *K maps $\mathfrak{M}$ into $\mathfrak{N}$.*

Let U and V be b-spaces and let K map U into V. We define $m(K)$ as the infimum, and $M(K)$ as the supremum, of the quantity $\|Ku_1-Ku_2\|/\|u_1-u_2\|$ taken over all u_1, u_2 in U with $u_1\neq u_2$. If $M(K)<\infty$, then we shall say that K is *bounded.* The space V is *complete* if each Cauchy sequence in V has a limit in V. If K is bounded and if, for each bounded set $\bigcup_\alpha u_\alpha$ in U, the set of images, $\bigcup_\alpha(Ku_\alpha)$, contains a sequence which converges to an element in V, then K is said to be *completely continuous.* We shall say that *V is complete for K* if, for each Cauchy sequence $\{u_n\}$ in U, the sequence $\{Ku_n\}$ has a limit in V.

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REMARK 1. It is easy to show that: (a) if V is a complete space and K is bounded, then V is complete for K; (b) if K is completely continuous, then V is complete for K.

We shall say that K is *closed* if the conditions $\lim \{u_n\} = u$ and $\lim \{Ku_n\} = v$ together imply that $Ku = v$. We denote by I_U the function defined, for each u in U, by $I_U(u) = u$. If K_1 and K_2 each map U into V, then $K_1 + K_2$ and $K_1 - K_2$ are mappings, defined for each u in U by the equations $(K_1+K_2)u = K_1u + K_2u$ and $(K_1 - K_2)u = K_1u - K_2u$.

We assume throughout this paper that X and Y are fixed b-spaces (not necessarily distinct), that F is a given mapping from X into Y, and that $m(F) = f > 0$.

3. **Solution of equations.** We consider the equation

$$Fx = y. \tag{1}$$

THEOREM 1. *If there exists a b-space Z, a mapping A from Y into Z and a mapping B from Z into X such that $m(A) = a > 0$, $M(R) = r < 1$, where $R = I_Z - AFB$, and if either of the following two conditions are satisfied* (c$_1$) *Z is complete for R or* (c$_2$) *X is complete for B, and F is closed; then, for each y in Y,* (i) *the sequence $\{x_n\}$ defined by $x_{n+1} = Bz_{n+1}$, where $z_{n+1} = Rz_n + Ay$ and z_0 is an arbitrary element of Z, converges, say to x,* (ii) *x is a unique solution of equation* (1), (iii) $\|x - x_n\| \leqq (r(1+r)/af(1-r))\|z_n - z_{n-1}\| \leqq (r^n(1+r)/af(1-r))\|z_1 - z_0\|$.

PROOF. We first note that $M(B) \leqq (1+r)/af$, since for any u and v in Z, $af\|Bu - Bv\| \leqq \|AFBu - AFBv\| = \|u - v + Rv - Ru\| \leqq (1+r) \cdot \|u - v\|$. Next $\|z_{n+1} - z_n\| = \|Rz_n - Rz_{n-1}\| \leqq r\|z_n - z_{n-1}\| \leqq \cdots \leqq r^n\|z_1 - z_0\|$. Hence $\|z_{n+k} - z_n\| \leqq \sum_{i=1}^{k}\|z_{n+i} - z_{n+i-1}\| \leqq (\sum_{i=1}^{k} r^i)\|z_n - z_{n-1}\| \leqq (r/(1-r))\|z_n - z_{n-1}\| \leqq (r^n/(1-r))\|z_1 - z_0\|$. Also $\|x_{n+k} - x_n\| = \|Bz_{n+k} - Bz_n\| \leqq (r(1+r)/af(1-r))\|z_n - z_{n-1}\| \leqq (r^n(1+r)/af(1-r)) \cdot \|z_1 - z_0\|$. Hence, the sequences $\{z_n\}$ and $\{x_n\}$ each converge in the sense of Cauchy. Now we verify (i). If we assume the hypothesis (c$_1$) then $\{Rz_n\}$ converges. But $z_{n+1} = Rz_n + Ay$, so that $\{z_n\}$ converges, say to z. Let $Bz = x$. Then $\|x - x_n\| = \|Bz - Bz_n\| \leqq M(B)\|z - z_n\|$ so that $\{x_n\}$ converges to x. Assuming (c$_2$), we see that $\{Bz_n\}$ converges, say to x. Since $x_n = Bz_n$, $\{x_n\}$ converges to x. We next verify (ii). There is at most one solution of equation (1) since $Fx' = Fx$ implies $0 = \|Fx' - Fx\| \geqq f\|x' - x\|$; i.e. $x' = x$. Assuming (c$_1$) we have seen that $\{z_n\}$ converges to z and $\{x_n\}$ converges to $x = Bz$. Hence for this case $a\|Fx - y\| \leqq \|AFBz - Ay\| = \|AFBz - z_{n+1} + Rz_n\| \leqq \|z - z_{n+1}\| + r\|z_n - z\|$, so that $Fx = y$. Assuming (c$_2$) we have $z_{n+1} - z_n = Ay - AFBz_n$ so that $a\|y - Fx_n\| \leqq \|z_{n+1} - z_n\|$. Hence $\{Fx_n\}$ approaches y while $\{x_n\}$

approaches x so that $Fx=y$. (iii) Let k approach infinity in the estimate above for $\|x_{n+k}-x_n\|$.

REMARK 2. The conclusion (iii) of Theorem 1 may be interpreted as follows: If an error of ϵ can be tolerated in the calculation of x, then the iteration may be stopped when

$$\|z_n-z_{n-1}\| \leqq af(1-r)\epsilon/r(1+r).$$

One can also be sure, after having computed only z_1, that the number of iterations required will not exceed

$$(\log\ [\epsilon af(1-r)]-\log\ [(1+r)\|z_1-z_0\|])/\log r.$$

If $\bar{x}$ is chosen arbitrarily in X as an approximation for x (e.g. an intuitive modification of an x_n), an estimate of the error is provided by the inequality $\|x-\bar{x}\| \leqq (1/f)\|y-F\bar{x}\|$. If we take $\bar{x}=\theta$ in the last inequality, then we have, in advance of the computation, an estimate of the magnitude of the solution sought.

REMARK 3. If the range of F is Y, then F has a right inverse mapping. Then one can choose this mapping as B, set $Z=Y$ and $A=I_Y$; the hypotheses of Theorem 1 (using c_1) will then be satisfied. Thus the quantities Z, A, B mentioned in the theorem will exist if, and only if, the range of F is Y.

REMARK 4. In virtue of Remark 1, it follows that the alternatives (c_1), (c_2) of the theorem can be replaced by the four alternatives: (c_1') Z is complete, (c_1'') R is completely continuous, (c_2') X is complete and F is closed, (c_2'') B is completely continuous and F is closed.

This remark explains the common basis of the two settings in which linear problems are treated.[1] Furthermore, it is not difficult to find a mapping such that V is not a complete space, K is not completely continuous, but V is complete for K.

REMARK 5. We present here a simple illustration of an application of the theorem, reserving a more extended study of applications for a subsequent paper. Let $\mathfrak{J}$ denote the interval $0 \leqq t \leqq T$ and $\mathfrak{H}$ the interval $-\infty<\tau<\infty$. Let C, D, and E each map $\mathfrak{H}\times\mathfrak{J}$ into $\mathfrak{H}$. Assume that each is continuous and that there are numbers c, d, e such that, for each t in $\mathfrak{J}$ and each τ and σ in $\mathfrak{H}$, $|C(\tau,t)-C(\sigma,t)| \leqq c|\tau-\sigma|$, $|D(\tau,\ t)-D(\sigma,\ t)| \leqq d|\tau-\sigma|$, $|E(\tau,\ t)-E(\sigma,\ t)| \leqq e|\tau-\sigma|$. Let $\mathfrak{C}^n$ denote the class of functions having a continuous nth derivative on $\mathfrak{J}$. Assume that $p \geqq 0$ and that $\alpha=c+dT+pT+eT^2<1$ and $\beta=2c+2dT+eT^2<1$. Assume also that $y(t)\in\mathfrak{C}^0$ (the 0th derivative of a

[1] Cf. A. T. Lonseth, *The propagation of error in linear problems*, Trans. Amer. Math. Soc. vol. 62 (1947) p. 194, lines 32–35.

function is the function itself). Let $\dot{x}$ denote dx/dt and $\ddot{x}$ denote d^2x/dt^2. Problem: Solve the equation

$$\ddot{x} + C(\ddot{x}, t) + D(\dot{x}, t) + p\dot{x} + E(x, t) = y(t) \tag{1'}$$

for $x(t)$ in $\mathfrak{C}^2$ with $x(0) = \dot{x}(0) = 0$.

To apply the theorem, let X be the subset of $\mathfrak{C}^2$ having the property that for each $\xi(t)$ in X, $\xi(0) = \dot{\xi}(0) = 0$. Let Y and Z be $\mathfrak{C}^0$. In each case let $\|g\| = \max_{t\in\mathfrak{J}}|g(t)|$. For each ξ in X let $F\xi = \ddot{\xi} + C(\ddot{\xi}, t) + D(\dot{\xi}, t) + p\dot{\xi} + E(\xi, t)$. One has for each u and v in X and each t in $\mathfrak{J}$, $|Fu - Fv| \geqq |\ddot{u} - \ddot{v}| - c\|\ddot{u} - \ddot{v}\| - (d+p)\|\dot{u} - \dot{v}\| - e\|u - v\|$, so that $\|Fu - Fv\| \geqq ((1-\alpha)/T^2)\|u - v\|$. For each ζ in $\mathfrak{C}^0$ let

$$B\zeta = \textstyle\int_0^t (t-s)\ \zeta(s)ds \text{ and } A\zeta = \zeta(t) - p\int_0^t e^{-p(t-s)}\zeta(s)ds.$$

Then $m(A) \geqq e^{-pT}$, and $M(R) \leqq \beta(1 - (e^{-pT}/2))$. Finally Z is a complete space (cf. Remark 4), so that all the conditions of the theorem are satisfied.

4. **Implicit functions.** Let $\mathfrak{T}$ be a non-null set and let G map $\mathfrak{T} \times X$ into Y. We now consider the equation

$$G(t, \mathfrak{X}) = \theta. \tag{2}$$

If there is one and only one function g, mapping $\mathfrak{T}$ into X, such that, for each t in $\mathfrak{T}$, $G(t, g(t)) = \theta$, then we shall say that *equation* (2) *defines the function* g.

For a fixed t in $\mathfrak{T}$, the function G_t defined, for each u in X, by $G_t u = G(t, u)$ maps X into Y. The following theorem is equivalent to Theorem 1.

THEOREM 1′. *Let G be given. If, for each fixed t in $\mathfrak{T}$, the hypotheses of Theorem* 1 *are satisfied* (*with F there replaced by G_t*) *then equation* (2) *defines a function g. For each fixed t in $\mathfrak{T}$ the computation of $\mathfrak{X} = g(t)$ may be carried out as in Theorem* 1, *namely $\mathfrak{X}_{n+1} = B\zeta_{n+1}$, where $\zeta_{n+1} = \zeta_n - AG(t, B\zeta_n) + A(\theta)$ and ζ_0 is an arbitrary element of Z. Estimates of the error corresonding to conclusion* (iii) *of Theorem* 1 *are valid.*

It is clear that Theorem 1 implies Theorem 1′. To verify the converse, let y be given. Let $\mathfrak{T}$ consist of the single element τ and define $G(\tau, x) = Fx - y$. Then if Z, A, and B exist satisfying the hypotheses of Theorem 1, define, for each u in Y, $A'u = A(u+y)$. Then $m(A') = m(A)$ and $A'G_\tau = AF$, so that the hypotheses of Theorem 1′ are satisfied with the available Z and B and with A replaced by A'. The conclusions of Theorem 1′ then imply the conclusions of Theorem 1.

THEOREM 2. *Let equation* (2) *define the function* g *and let* $\mathfrak{X}=g(t)$. *Let* t_0 *be a fixed element of* $\mathfrak{T}$ *and* $\mathfrak{X}_0=g(t_0)$. *If* $\mathfrak{T}$ *is a space in which convergence is defined, and if whenever* t *approaches* t_0, $\|G(t, \mathfrak{X}_0)\|/m(G_t)$ *approaches zero, then* g *is continuous at* t_0; *in particular if* $\mathfrak{T}$ *is a neighborhood space and if there is a neighborhood* N *of* t_0 *such that* $\inf_{t\in N} m(G_t) >0$ *and if the function* $G_{\mathfrak{X}_0}$ *from* $\mathfrak{T}$ *into* Y *defined for each* t *in* $\mathfrak{T}$ *by* $G_{\mathfrak{X}_0}(t)=G(t, \mathfrak{X}_0)$ *is continuous at* t_0, *then* g *is continuous at* t_0.

PROOF.

$$\begin{aligned}\|\mathfrak{X}_0 - \mathfrak{X}\| &\leq \frac{1}{m(G_t)}\|G_t\mathfrak{X}_0 - G_tg(t)\| \\ &= \frac{\|G(t, \mathfrak{X}_0)\|}{m(G_t)} = \frac{\|G(t, \mathfrak{X}_0) - G(t_0, \mathfrak{X}_0)\|}{m(G_t)}.\end{aligned}$$

5. **Perturbed equations.** Let Φ map X into Y. We now consider the equation

$$\Phi\xi = \eta. \tag{3}$$

Suppose Φ is unavailable in a form suitable for computations, while F, which is considered more suitable for computations, is an approximation for Φ. If also the exact value of η is not available, but y is a convenient approximation for η, then we propose to use the solution x of equation (1) as an approximation for the solution, ξ, of equation (3). If we let $\phi=F-\Phi$, $\beta=y-\eta$, $\alpha=x-\xi$ we see that we shall solve the "perturbed" equation

$$(\Phi + \phi)(\xi + \alpha) = \eta + \beta. \tag{4}$$

Since it may be useful to know a bound for the error α before the computation of x is made, it is desirable to present a bound which is independent of x.

CRITERION 1. *Let* F *be given and let* y *be a given element of* Y. *Assume that* $M(\phi)=\delta<f$. *If there exists an* x *satisfying equation* (1), *then this solution is unique and*

$$\|\alpha\| \leq \frac{\|\beta\| + \|\phi x\|}{f - \delta} \leq \frac{f(\|\beta\| + \|\phi(\theta)\|) + \delta\|y - F(\theta)\|}{f(f - \delta)}.$$

PROOF. The condition $f>0$ implies the uniqueness. Also $f\|\alpha\| \leq \|Fx-F\xi\| = \|\beta-\phi\xi\| \leq \|\beta\|+\delta\|\alpha\|+\|\phi x\|$.

6. **Output errors.** We might, as an alternative to the viewpoint of §5, regard equation (3) as defining η in terms of Φ and ξ. Suppose, due to previous errors in the computations, we have available, instead

of Φ and ξ, approximations to them, F and x. Then we propose to use $y=Fx$ as an approximation for η. As above let $\phi=F-\Phi$, $\alpha=x-\xi$, $\beta=y-\eta$.

CRITERION 2. *Let F and x be given. If $M(\phi)=\delta$, $M(F)=C$, then* $\|\beta\| \leqq (C+\delta)\|\alpha\|+\|\phi x\| \leqq (C+\delta)\|\alpha\|+\|\phi(\theta)\|+\delta\|x\|$.

PROOF. $\beta=Fx-\Phi\xi=(Fx-F\xi)+(\phi\xi-\phi x)+\phi x$.

7. **Characteristic value problem.** Let W be a set of elements satisfying the following conditions: (h_1) *W is a metric space with metric function ρ*; (h_2) *for each real (or complex) number γ and each w in W, γw is in W*; (h_3) *W has a center of homogeneity, ϕ, i.e. for each number γ and each w in W, $\rho(\gamma w, \phi)=|\gamma|\rho(w, \phi)$*; ($h_4$) *if $\{\gamma_n\}$ approaches γ and $\{w_n\}$ approaches w and $\{\gamma_n w_n\}$ approaches w', then $w'=\gamma w$.* Denote $\rho(w, \phi)$ by $L(w)$. We shall write $\gamma^{-1}w$ as $(1/\gamma)w$ or w/γ.

Let p be a positive number and let S_p denote the set of all w in W such that $L(w)=p$. Let H map S_p into W. If there is a ψ in S_p and a number $\lambda\neq 0$ such that $(1/\lambda)H\psi=\psi$, then we say that λ is a *(p)-characteristic value* having the *(p)-characteristic vector* ψ. We assume that, for each w in S_p, $Hw\neq\phi$ and that H is continuous on S_p.

Take any ψ_0 in S_p. Let $\psi_{n+1}=(1/\lambda_n)H\psi_n$, where $\lambda_n=L(H\psi_n)/p$ $(n=0, 1, 2, \cdots)$. If $\{\psi_n\}$ converges, say to ψ, then, clearly, $\{H\psi_n\}$ converges to $H\psi$, $\{\lambda_n\}$ converges, say to λ. Then λ is a (p)-characteristic value having the (p)-characteristic vector ψ. Here $\lambda>0$ but if one uses instead $\lambda_n=-L(H\psi_n)/p$, then $\lambda<0$. (In the complex case we may take $\lambda_n=e^{i\omega}L(H\psi_n)/p$.) We postpone to a later paper a study of the convergence of $\{\psi_n\}$, limiting ourselves here to one situation, for which the convergence will be demonstrated. We shall not require that $(\alpha\beta)w=\alpha(\beta w)$.

We shall now replace condition (h_4) by: (h_5) *if α, β, and γ are numbers and u and v are in W, then $\rho(\gamma u, \gamma v)=|\gamma|\rho(u, v)$ and $\rho(\alpha u, \beta u)=|\alpha-\beta|L(u)$.* From this it follows that if $\{\gamma_n\}$ converges to γ and $\{w_n\}$ converges to w, then $\{\gamma_n w_n\}$ converges to γw, since $\rho(\gamma w, \gamma_n w_n) \leqq \rho(\gamma w, \gamma w_n)+\rho(\gamma w_n, \gamma_n w_n) \leqq |\gamma|\rho(w, w_n)+|\gamma-\gamma_n| \cdot(L(w)+\rho(w_n, w))$.

THEOREM 3. *Let W satisfy* (h_1), (h_2), *and* (h_5) *over the complex numbers. Let H map S_p into W. We assume that there is a number $M>0$ such that, for each u, v in S_p, $\rho(Hu, Hv) \leqq M\rho(u, v)$. Let W be complete for H (for each Cauchy sequence $\{v_n\}$ in S_p, $\{Hv_n\}$ has a limit in W). Let $\inf_{v\in S_p}L(Hv)=c$. We assume that $q=2Mp/c<1$. Let ω be any real number. Then the sequences $\{\psi_n\}$, $\{\lambda_n\}$ defined by $\psi_{n+1}=H\psi_n/\lambda_n$, $\lambda_n=e^{i\omega}L(H\psi_n)/p$ $(n=0, 1, 2, \cdots)$, ψ_0 an arbitrary element of S_p, converge respectively to ψ (a (p)-characteristic vector) and λ (the (p)-char-*

acteristic value for ψ); λ *has argument* ω. *We have the estimates of error*

$$\rho(\psi, \psi_n) \leqq \frac{q}{1-q}\rho(\psi_n, \psi_{n-1}) \leqq \frac{q^n}{1-q}\rho(\psi_1, \psi_0),$$

$$|\lambda - \lambda_n| = \frac{1}{p}|L(H\psi) - L(H\psi_n)| \leqq \frac{1}{p}\rho(H\psi, H\psi_n) \leqq \frac{M}{p}\rho(\psi, \psi_n).$$

PROOF. We have

$$\begin{aligned}
\rho(\psi_{n+1}, \psi_n) &= \rho\left(\frac{H\psi_n}{\lambda_n}, \frac{H\psi_{n-1}}{\lambda_{n-1}}\right) \\
&\leqq \rho\left(\frac{H\psi_n}{\lambda_n}, \frac{H\psi_{n-1}}{\lambda_n}\right) + \left|\frac{1}{\lambda_n} - \frac{1}{\lambda_{n-1}}\right| L(H\psi_{n-1}) \\
&= \frac{1}{|\lambda_n|}(\rho(H\psi_n, H\psi_{n-1}) + |L(H\psi_{n-1}) - L(H\psi_n)|) \\
&\leqq \frac{2}{|\lambda_n|}\rho(H\psi_n, H\psi_{n-1}) \leqq q\rho(\psi_n, \psi_{n-1}),
\end{aligned}$$

since $|\lambda_n| \geqq c/p$. Thus (as in the proof of Theorem 1) $\rho(\psi_{n+k}, \psi_n) \leqq (q^n/(1-q))\rho(\psi_1, \psi_0)$, and $\{\psi_n\}$ is a Cauchy sequence in S_p. Since W is complete for H, $\{H\psi_n\}$ converges, says to w. Then $\{\lambda_n\}$ converges to $e^{i\omega}L(w)/p = \lambda$. Hence $\{\psi_n\}$ also converges.

REMARK 6. Under the conditions of the theorem we can show the nondegeneracy, continuity, and boundedness of quantities which in the associative case are the eigenvalues.

(a) If $H\psi = \lambda\psi$ and $Hv = \lambda v$ (or if $H\psi/\lambda = \psi$ and $Hv/\lambda = v$), then $\psi = v$.

PROOF. $M\rho(\psi, v) \geqq \rho(H\psi, Hv) = |\lambda| \cdot \rho(\psi, v) \geqq 2M\rho(\psi, v)$.

(b) If $H\psi = \lambda\psi$ and $Hu = \mu u$, then

$$\frac{|\lambda| + M}{p}\rho(\psi, u) \geqq |\lambda - \mu| \geqq \frac{|\lambda| - M}{p}\rho(\psi, u).$$

PROOF.

$$\begin{aligned}
M\rho(\psi, u) &\geqq |\rho(H\psi, \lambda u) - \rho(\lambda u, Hu)| \\
&= ||\lambda| \cdot \rho(\psi, u) - |\lambda - \mu| \cdot L(u)|.
\end{aligned}$$

(c) If $H\psi = \lambda\psi$ and $Hu = \mu u$, then $0 < |\lambda| - 2M \leqq |\mu| \leqq |\lambda| + 2M$.

PROOF. $2pM \geqq M\rho(\psi, u) \geqq \rho(H\psi, Hu) \geqq p||\lambda| - |\mu||$.

REMARK 7. As an illustration of a situation satisfying the conditions of the theorem let x_0 be a fixed element of a Banach space $\mathfrak{X}$

with $\|x_0\|=4$. For each x in $\mathfrak{X}$ let $Hx=x_0+x$. Then, with $p=1$ we have $M=1$, $c=3$, $q=2/3$.

REMARK 8. If, in addition to the hypotheses of the theorem, H maps S_p onto a subset of S_k, then $c=k$ and $|\lambda_n|=k/p$. From the inequality $\rho(\psi_{n+1}, \psi_n)=(p/k)\rho(H\psi_n, H\psi_{n-1})\leqq(pM/k)\rho(\psi_n, \psi_{n-1})$, we see that we can replace the condition $q=2Mp/c<1$ of Theorem 3 by the condition $q'=Mp/k<1$ and, in the estimates of error, the q may be replaced by q'.

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