Chevalley and Schafer \cite{4}\textsuperscript{2} have shown that the exceptional simple Lie algebra \( F_4 \) of dimension 52 over an arbitrary algebraically closed field \( \Omega \) of characteristic 0 is the derivation algebra of the unique exceptional simple Jordan algebra of dimension 27 over \( \Omega \). In this paper we show that a Lie algebra \( \mathfrak{g} \) over an arbitrary field \( \Phi \) of characteristic 0 is of type \( F \) if and only if \( \mathfrak{g} \) is isomorphic to the derivation algebra \( \mathfrak{D}(\mathfrak{J}) \) of an exceptional central simple Jordan algebra \( \mathfrak{J} \) over \( \Phi \). The proof given for this theorem requires a characterization of the automorphisms of \( \mathfrak{D}(\mathfrak{J}) \) over \( \Omega \). We prove that every automorphism of \( \mathfrak{D}(\mathfrak{J}) \) has the form \( D \to SDS^{-1} \) for a unique automorphism \( S \) of \( \mathfrak{J} \). The classification of Lie algebras of type \( F \) over \( \Phi \) is reduced to the problem of classifying exceptional central simple Jordan algebras over \( \Phi \), since it is shown that \( \mathfrak{D}(\mathfrak{J}_1) \approx \mathfrak{D}(\mathfrak{J}_2) \) if and only if \( \mathfrak{J}_1 \cong \mathfrak{J}_2 \). In the last section of this paper the three exceptional central simple Jordan algebras over a real closed field are exhibited and their derivation algebras are the real closed Lie algebras of type \( F \).

1. Exceptional central simple Jordan algebras. Let \( \Omega \) be an algebraically closed field of characteristic 0. The exceptional simple Jordan algebra \( \mathfrak{J} \) over \( \Omega \) is the nonassociative algebra of dimension 27 whose elements are \( 3 \times 3 \) Hermitian matrices with elements in the unique Cayley algebra \( \mathfrak{C} \) of dimension 8 over \( \Omega \). Thus the elements of \( \mathfrak{J} \) have the form

\[
x = \begin{pmatrix}
\xi_1 & c_2 & \bar{c}_3 \\
\bar{c}_2 & \xi_2 & c_1 \\
c_2 & \bar{c}_1 & \xi_1
\end{pmatrix},
\]

\( \xi_i \) in \( \Omega \) and \( c_i, \bar{c}_i \) in \( \mathfrak{C} \) \((i = 1, 2, 3)\) where \( \bar{c}_i \) is the conjugate of \( c_i \) \cite[p. 83]{8}. Multiplication in \( \mathfrak{J} \) is defined as \( xy = (x \circ y + y \circ x)/2 \) where \( x \circ y \) is the ordinary matrix product. Let \( \mathfrak{E}_i \) be the matrix with \( \xi_i = 1 \), all other entries 0, and \( \mathfrak{E}_i \) be the set of matrices with all entries
0 except \(c_i\) and \(\xi_i\) \((i = 1, 2, 3)\). In this section \(i, j, k\) will be a permutation of \(1, 2, 3\). Elements of \(X_i\) will be denoted by \(t_i\) or \(t'_i\). Clearly multiplication is commutative and is such that; \(e_i^2 = e_i, e_i e_j = 0, e_i e_i = 0, e_i t_i = t_i/2, t_i t'_i \) in \(\Omega(e_j + e_k)\), and \(t_i t'_i\) in \(X_i\).

The derivation algebra \(\mathfrak{d}(\mathfrak{F})\) of \(\mathfrak{F}\) is the Lie algebra of linear transformations \(D\) on \(\mathfrak{F}\) satisfying

\[D(xy) = (Dx)y + x(Dy)\].

Chevalley and Schafer have proved \([4]\) that \(\mathfrak{d}(\mathfrak{F})\) is the exceptional simple Lie algebra \(F_4\) of dimension 52 and rank 4 over \(\Omega\). Let \(\mathfrak{d}_0\) be the subalgebra of derivations which map \(e_1, e_2,\) and \(e_3\) into 0, and \(\mathfrak{d}_i\) the subalgebra which maps \(e_i\) into 0. Since \(D_1 = 0, D_i e_j = -D_j e_k\) for \(D_i\) in \(\mathfrak{d}_i\). In \([4]\) it is shown that for any \(D_0\) in \(\mathfrak{d}_0, D_0 t_i\) is in \(\mathfrak{d}_i\), and that for any \(t_i\) in \(X_i\) there is a \(D_i\) in \(\mathfrak{d}_i\) such that \(D_i t_i = t_i\).

Associated with the algebra \(\mathfrak{F}\) is a symmetric nondegenerate bilinear form \(Sp xy\) where \(Sp x = \xi_1 + \xi_2 + \xi_3\). This bilinear form is left invariant by any derivation, i.e. \(Sp (Dx)y + Sp x(Dy) = 0\). The set of elements \(x\) such that \(Sp x = 0\) form a subspace \(\mathfrak{F}_0\) of \(\mathfrak{F}\) of dimension 26. \(\mathfrak{F}_0\) is an irreducible representation space of \(\mathfrak{d} = \mathfrak{d}(\mathfrak{F})\). We denote the restriction of \(D\) to \(\mathfrak{F}_0\) also by \(D\). If \(R\) is any linear transformation on \(\mathfrak{F}_0\) which commutes with all \(D\) in \(\mathfrak{d}\), then \(R = \sigma I\), for by Schur's Lemma the set of linear transformations which commute with \(\mathfrak{d}\) form a division algebra containing \(I\) (the identity linear transformation on \(\mathfrak{F}_0\)) and since \(\Omega\) is algebraically closed this set is \(\Omega I\).

Since \(Sp xy\) is a nondegenerate bilinear form we may define the adjoints \(A^*\) of any linear transformation \(A\) on \(\mathfrak{F}\) by

\[Sp (A^*x)y = Sp x(Ay), \quad \text{for all } x, y \in \mathfrak{F}\].

The restriction of \(Sp xy\) to \(\mathfrak{F}_0 \times \mathfrak{F}_0\) is also a nondegenerate bilinear form. For any linear transformation \(B\) on \(\mathfrak{F}_0\) we similarly define the adjoint \(B^*\), on \(\mathfrak{F}_0\), of \(B\). Since \(Sp (Dx_0)y_0 = -Sp x_0(Dy_0)\), for all \(x_0, y_0\) in \(\mathfrak{F}_0\), \(D^* = -D\) for any \(D\) in \(\mathfrak{d}\). The mapping \(A \rightarrow A^*\) is an involutorial anti-automorphism in the algebra of linear transformations on \(\mathfrak{F}_0\).

Let \(\Phi\) be a field of characteristic 0. The exceptional simple Jordan algebras over \(\Phi\) are those simple Jordan algebras which are of degree 3 and dimension 27 over their centers. Exceptional central simple Jordan algebras over \(\Phi\) have been characterized by Schafer \([8]\). They are the algebras \(\mathfrak{J} = \mathfrak{J}(\mathfrak{C}, p)\) of \(3 \times 3\) matrices \(x\) with elements in a Cayley algebra \(\mathfrak{C}\) over \(\Phi\) satisfying \(x = p x^t p^{-1}\), \(p\) a nonsingular diagonal matrix in \(\Phi\) and \(x^t\) the conjugate transpose of \(x\). Multiplication in \(\mathfrak{J}\) is defined by \(xy = (x \circ y + y \circ x)/2\), where \(x \circ y\) is the...
ordinary matrix multiplication. For x, y, z in \mathfrak{F} the associator
A(x, y, z) is defined as \( A(x, y, z) = (xy)z - x(yz) \). The subspace \mathfrak{P}
spanned by all associators is called the associator subspace of \mathfrak{F},
and it is known [9] that \mathfrak{F} is the direct sum \( \mathfrak{F} = \mathfrak{P} + \mathfrak{F} \). Since
Sp \((xy)z\) = Sp \(x(yz)\) for \(x, y, z\) in \mathfrak{F}, we have Sp \(x = 0\) for all \(x\) in \mathfrak{P}.
From the direct sum decomposition of \mathfrak{F}, \mathfrak{P} has dimension 26 and
hence \mathfrak{F} is the set \mathfrak{F}_0 of all \(x\) in \mathfrak{F} for which Sp \(x = 0\).

From this characterization of \mathfrak{F}_0, it is easy to see that \(\mathfrak{F}_0 = \mathfrak{F}_0\)
for any automorphism \(S\) of \mathfrak{F}, since \mathfrak{F}_0 is spanned by the elements
\(A(Sx, Sy, Sz)\). Also \((J_0)_2 = (J_2)_0\) for any extension \(\Sigma\) of \(\Phi\).

It is known that every derivation \(D\) of \(\mathfrak{F}\) is inner; that is, \(D\) has
the form \(y \rightarrow Dy = \Sigma A(x, y, z)\) [7, Theorem 2]. Hence \(\mathfrak{F} = \mathfrak{F}_0\). More-
over \(\mathfrak{F}_0\) is an irreducible representation space for \(\mathfrak{F}\). For if \(\mathfrak{M}\) is
invariant with respect to \(\mathfrak{D}\), then \(\mathfrak{M}_0\) is invariant with respect to \(\mathfrak{D}_0\).
But it is known [4, p. 141] that \(\mathfrak{F}_0\) is an irreducible representation
space for \(\mathfrak{D}_0\), \(\Omega\) the algebraic closure of \(\Phi\).

2. Similarity of representations of \(F_4\). Let \(\Omega\) be an arbitrary algebra-
ically closed field of characteristic 0. \(F_4\) is the exceptional simple
Lie algebra of dimension 52 over \(\Omega\). In [2] it is shown that if \(\Lambda = m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4\) is a weight of a representation \(\mathcal{P}\) of \(F_4\),
then \(2m_i, m_i \pm m_j, \) and \(m_1 \pm m_2 \pm m_3 \pm m_4\) are integers and the linear
forms:

\[(1) \quad \Lambda - \lambda_i, \Lambda - 2\lambda_i, \cdots, \Lambda - 2m_1\lambda_4,\]
\[(2) \quad \Lambda - (m_i \pm m_j), \cdots, \Lambda - (m_i \pm m_j)(\lambda_i \pm \lambda_j),\]
\[(3) \quad \Lambda - (\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)/2, \cdots, \]
\[\Lambda - ((m_i \pm m_2 \pm m_3 \pm m_4)/2)(\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)\]
\((i, j = 1, \cdots, 4)\) are also weights of \(\mathcal{P}\). From (1) it can be seen that
if \(\Lambda\) is a weight then so is \(-\Lambda\), from (2) that if \(\Lambda\) is a weight then so is
\(\Lambda'\) where \(\Lambda'\) is obtained from \(\Lambda\) by a permutation of the \(m_i\), and (1),
(2), and (3) give that for a highest weight:

\[(4) \quad m_1 \geq m_2 \geq m_3 \geq m_4 \geq 0,\]
\[(5) \quad m_1 \geq m_2 + m_3 + m_4.\]

It is shown in [4] that there is an irreducible representation of \(F_4\) of
degree 26.

**Lemma.** Any two irreducible representations of degree 26 of the Lie
algebra \(F_4\) are similar.

For the proof it is sufficient to show [2] that any two irreducible
representations of $F_4$ of degree 26 have the same highest weight. Cartan has also shown in [2] that the number of weights of a representation does not exceed the degree of the representation. From (4) and (5) it may be seen that if $\Lambda$ is the highest weight then $\Lambda$ is in one of the following:

(i) At least three distinct $m_i > 0$,
(ii) $m_1 > m_2 = m_3 = m_4 > 0$,
(iii) $m_1 = m_2 > m_3 = m_4 = 0$,
(iv) $m_1 > m_2 = m_3 = m_4 = 0$,
(v) $m_1 = m_2 = m_3 = m_4 = 0$.

Note that (v) would imply that the representation is zero and therefore reducible, which is a contradiction. Cases (i), (ii), and (iii) may be eliminated, for by using the properties of weights of $F_4$, as given above, it may be seen that in these cases there would be more than 26 distinct weights. In (iv), $\Lambda = m\lambda_1$, $m$ a positive integer. If $m \geq 2$ then there are again more than 26 distinct weights, thus the only possible highest weight is $\Lambda = \lambda_1$.

3. Automorphisms of $F_4$. This section is devoted to the proof of the following theorem which characterizes the automorphisms of the exceptional Lie algebra $F_4$ of dimension 52 over $\Omega$.

THEOREM 1. If $D \rightarrow D^g$ is an automorphism of $\mathfrak{D}(\mathfrak{F})$, $\mathfrak{F}$ the exceptional simple Jordan algebra over an algebraically closed field $\Omega$ of characteristic 0, then there is a unique automorphism $S$ of $\mathfrak{F}$ such that $D^g = SDS^{-1}$.

The automorphism $S$ defines a second irreducible representation of the Lie algebra $\mathfrak{D} = \mathfrak{D}(\mathfrak{F})$ acting on $\mathfrak{F}_0$. By the lemma of §2 there is a nonsingular linear transformation $S_1$ on $\mathfrak{F}_0$ such that for $D$ and $D^g$ on $\mathfrak{F}_0$, $D^g = S_1DS_1^{-1}$. Then $D^g = (S_1^*)^{-1}DS_1^*$ or $D = S_1^*S_1D(S_1^*S_1)^{-1}$. Thus $S_1^*S_1$ commutes with every $D$ in $\mathfrak{D}$ and $S_1^*S_1 = \sigma I$, $\sigma \neq 0$ in $\Omega$. Let $S_2 = \sigma^{-1/2}S_1$, where we reserve until later the choice of which square root of $\sigma$ we use. Define a linear transformation $S$ on $\mathfrak{F}$ as follows:

$$Sx = S(\gamma_1 + x_0) = \gamma_1 + S_2x_0, \quad \gamma \in \Omega, \quad x_0 \in \mathfrak{F}_0.$$ 

$S$ is a nonsingular linear transformation on $\mathfrak{F}$ and the mapping $D \rightarrow SDS^{-1}$ of $\mathfrak{D}$ is the same as $D \rightarrow D^g$. It is easily seen that $S^*x = S^*(\gamma_1 + x_0) = \gamma_1 + S_2^*x_0$ for all $x$ in $\mathfrak{F}$. Hence $S^*S = I$, the identity on $\mathfrak{F}$, and

$$\text{Sp } (Sx)(Sy) = \text{Sp } (S^*Sx)y = \text{Sp } xy. \quad (6)$$
Furthermore since $S1 = 1$,

(7) $\text{Sp} Sx = \text{Sp} (Sx)(S1) = \text{Sp} x$.

We shall show that $S$ is an automorphism of $\mathfrak{g}$. Let $i$, $j$, $k$ be a permutation of 1, 2, 3 for the remainder of this section. We begin by showing that the elements $f_i = S e_i$ are pairwise orthogonal idempotents of $\mathfrak{g}$. We write $u_i = St_i$. Since $S$ is a nonsingular linear transformation on $\mathfrak{g}$, any element $y$ of $\mathfrak{g}$ may be written uniquely as $y = \alpha f_1 + \alpha f_2 + \alpha f_3 + u_1 + u_2 + u_3$. Let

(8) $f_i^2 = \alpha_i f_i + \alpha_i f_j + \alpha_i e_k + u_i^{(i)} + u_j^{(i)} + u_k^{(i)}$.

For any $D_0$ in $\mathfrak{d}_0$ we apply the derivation $D_0^{\mathfrak{f}}$ to (8) to obtain

\[
2f_i(D_0 f_i) = 2f_i(SD_0 e_i) = 0 = SD_0(\alpha_i f_i + \alpha_i f_j + \alpha_i e_k + u_i^{(i)} + u_j^{(i)} + u_k^{(i)}) = SD_0(u_i^{(i)} + u_j^{(i)} + u_k^{(i)}).
\]

Since $S$ is nonsingular:

\[
D_0(u_i^{(i)} + u_j^{(i)} + u_k^{(i)}) = 0.
\]

For all $D_0$ in $\mathfrak{d}_0$, $D_0 e_k$ is in $\mathfrak{d}_k$, $k = 1, 2, 3$, hence $D_0 e_k^0 = 0$ for any $D_0$ in $\mathfrak{d}_0$. Now $\mathfrak{d}_0$ on $\mathfrak{d}_k$ is the orthogonal Lie algebra $\mathfrak{o}(8, \Omega)$ [4]. The associative enveloping algebra of $\mathfrak{o}(8, \Omega)$ on a space of dimension 8 is the full matrix algebra $\Omega_8$. Hence $\alpha_i^0 = 0$ and $u_k^0 = 0$. Thus we have

(9) $f_i^2 = \alpha_i f_i + \alpha_i f_j + \alpha_i e_k$.

There exists a $D_i$ in $\mathfrak{d}_i$ such that $D_i$ is not in $\mathfrak{d}_0$ [4, p. 140]. For this $D_i$, $D_i e_i \neq 0$, for otherwise $D_i e_k = 0$ and $D_i$ is in $\mathfrak{d}_0$. For this $D_i$ we apply the derivation $D_i^{\mathfrak{f}} = SD_0 S^{-1}$ to (9) and get $0 = (\alpha_i - \alpha_i)SD_0 e_i$ or $\alpha_i = \alpha_i$. Equation (9) may be written as

(10) $f_i^2 = \alpha_i f_i + \beta_i f_j + \gamma_i f_k$.

Let $x$ be an element of $\mathfrak{g}$. By computing $x^2$ and $x^3$ it may be seen that $x$ satisfies the relation

(11) $x^2 + \alpha(x) x^2 + \beta(x) x + \gamma(x) 1 = 0$

where $\alpha(x) = -(\xi_1 + \xi_2 + \xi_3)$, $\beta(x) = \sum \xi_i c_j - \sum \pi(c_i)$, and $\gamma(x) = \sum \xi_i c_i c_j - \xi_i c_j c_k - \text{Sp} (c_i c_j c_k)$. Using the expressions given in [4] for $\text{Sp} x$, $\text{Sp} x^2$, and $\text{Sp} x^3$, we get $\alpha(x) = -\text{Sp} x$, $\beta(x) = \{ (\text{Sp} x)^2 - \text{Sp} x^2 \}/2$, and $\gamma(x) = \{3(\text{Sp} x)(\text{Sp} x^2) - (\text{Sp} x)^3 - 2\text{Sp} x^3 \}/6$. We compute the
coefficients in (11) for $f_i$. Equations (6) and (7) imply $Sp f_i = Sp e_i = 1$ and $Sp f_i^2 = Sp e_i^2 = Sp e_i = 1$, (10) gives $\alpha_i + 2\beta_i = 1$, and (10) may be rewritten as

\[(12) \quad f_i^2 = \beta_i 1 + (1 - 3\beta_i)f_i.\]

Hence

\[f_i^2 = \beta_i f_i + (1 - 3\beta_i)f_i\]

\[(13) = (\beta_i - 3\beta_i^2)1 + (1 - 5\beta_i + 9\beta_i^2)f_i.\]

Moreover, $Sp f_i^2 = 1 - 2\beta_i$. Substituting in (11),

\[(14) \quad f_i - f_i^2 = (2/3)\beta_i = 0.\]

Combining (12), (13), and (14) we get

\[((2/3)\beta_i - 3\beta_i^2)1 + (9\beta_i^2 - 2\beta_i)f_i = 0,\]

but $1$ and $f_i$ being linearly independent, $9\beta_i^2 - 2\beta_i = 0$. Thus $\beta_i = 0$ or $\beta_i = 2/9$. $Sx$ was defined as $S(\gamma_1 + x_0) = \gamma_1 + \sigma^{-1/2}S_1 x_0$ for $x = \gamma_1 + x_0$. Let $S'x = \gamma_1 - \sigma^{-1/2}S_1 x_0$. Then $S'$ has all the properties we have derived for $S$. $Sp x = Sp (Sx) = 3\gamma$ and $Sx + S'x = 2\gamma_1$ imply $Sx + S'x = (2/3)(Sp x)$. Calling $S'e_i = f_i'$, we have $f_i' = (2/3)1 - f_i$, and

\[(f_i')^2 = \beta_i 1 + (1 - 3\beta_i)f_i' = (\beta_i + 2/3)f_i' + (-1 + 3\beta_i)f_i = (2/3)1 - f_i^2 = (\beta_i + 4/9)1 + (-3\beta_i - 1/3)f_i.\]

Comparing coefficients we get $\beta_i' = 2/9 - \beta_i$. Thus if $\beta_i = 2/9$, $\beta_i' = 0$; and if $\beta_i = 0$, $\beta_i' = 2/9$. Therefore (by replacing $S$ by $S'$, if necessary) we may assume either that all $\beta_i = 0$ or that exactly one $\beta_i = 0$ (and $\beta_j = \beta_k = 2/9$). We shall show that the second case leads to a contradiction.

For any $t_i$ in $X_i$ there is a $D_i$ in $D_i$ such that $D_i e_j = t_i$. Then for this $D_i$, $SD_iS^{-1}f_i = S_t$. Apply $SD_iS^{-1}$ to the equation obtained from (12) by replacing $i$ by $j$:

\[(15) \quad 2f_j u_i = (1 - 3\beta_i)u_i.\]

Since $1u_i = (f_i + f_j + f_k)u_i$, if we use (15) together with (15) in which $j$ is replaced by $k$ we have

\[(16) \quad f_i u_i = (3\beta_i + 3\beta_k)u_i / 2.\]

If $\beta_i = 0$ and $\beta_j = \beta_k = 2/9$, $f_i$ is an idempotent and $f_i u_i = (2/3)u_i$, but
[1, p. 550] the only characteristic roots of $R_f$ are 0, 1/2, and 1. Since we may choose $u_i = St_i \neq 0$, we have a contradiction. Thus $\beta_i = 0$ for all $i$, and the elements $f_i$ are idempotents in $\mathfrak{S}$. Since $(f_i + f_j)^2 = (1 - f_k)^2$, we have $f_i f_j = 0$, or the $f_i$ are pairwise orthogonal.

Since $\beta_j + \beta_k = 0$, equations (15) and (16) imply that $f_i u_i = u_i / 2$, $f_i u_i = 0$ for any $u_i = St_i$ in $S \mathfrak{S}_i$. Hence

$$ (Se_i)^2 = S(e_i)^2 = Se_i, \quad (Se_i)(Se_j) = S(e_i e_j) = 0, \quad (Se_i)(St_i) = S(e_i t_i) = 0, \quad (Se_i)(St_j) = S(e_i t_j) = St_i / 2. $$

In order to show that $S$ is an automorphism of $\mathfrak{S}$, it remains only to show that

$$ (St_i)(St'_j) = S(t_i t'_j), \quad (St_i)(St_j) = S(t_i t_j). $$

We compute the product $t_i t'_j$ in $\Omega(e_j + e_k)$ for any $t_i$ and $t'_j$ in $\mathfrak{S}_i$ as follows: there is a $D_i$ in $\mathfrak{D}_i$ such that $D_i e_j = t_i$. Also [4, p. 140]

$$ D_i t'_j = \theta(e_j - e_k) + t_i, \quad \text{for } \theta \in \Omega, t_i' \in \mathfrak{S}_i. $$

Apply this $D_i$ to $e_j t'_j = t'_j / 2$ to obtain

$$ t_i t'_j = - \theta(e_j + e_k) / 2 $$

for $\theta$ in (19). To compute the product $(Se_i)(St'_j)$ we apply the derivation $SD_i S^{-1}$ to $(Se_j)(St'_j) = St'_j / 2$. Using (17), we obtain

$$ (St_i)(St'_j) = - \theta(S e_j + S e_k) / 2 $$

for $\theta$ in (19). Hence the first of equations (18) holds. Finally, given $t_i$ in $\mathfrak{S}_i$, $t_j$ in $\mathfrak{S}_j$, apply the derivation $D_i$ in $\mathfrak{D}_i$ satisfying $D_i e_j = t_i$ to the equation $e_i t_j = 0$. Since

$$ (Se_i)(St_j) = S(e_i t_j) = S(- S e_i e_j), $$

[4, p. 140], this gives $t_i t'_j = t'_j / 2$ in (22). To compute the product $(St_i)(St'_j)$ we apply $SD_i S^{-1}$ to $(Se_i)(St'_j) = 0$ in (17) to obtain $(St_i)(St'_j) = - St'_j / 2$ for $t'_j$ in (22). Hence $(St_i)(St_j) = S(t_i t_j)$.

It has been shown that $S$ is an automorphism of $\mathfrak{S}$; it remains to show that it is unique. Let $R$ be an automorphism of $\mathfrak{S}$ such that $D \to D^S = SDS^{-1} = RDR^{-1}$. In §1 we saw that $R \mathfrak{S}_0 = \mathfrak{S}_0$. Thus on $\mathfrak{S}_0$, $R^{-1} S$ commutes with all $D$. Hence $R^{-1} S = \sigma I$, $\sigma \neq 0$ in $\Omega$, on $\mathfrak{S}_0$; that is, $S = \sigma R$ on $\mathfrak{S}_0$. Choose $t_1$ in $\mathfrak{S}_1$, $t_2$ in $\mathfrak{S}_2$ such that $t_1 t_2 = t_2 \neq 0$ in $\mathfrak{S}_2$. $S t_1 = \sigma R t_1$, $S t_2 = \sigma R t_2$, and $S t_3 = (St_1)(St_2) = \sigma^2 (R t_1)(R t_2) = \sigma^2 R t_3$. Hence $\sigma = 1$. Since $R 1 = S 1$, $R = S$ on $\mathfrak{S}$.

4. Lie algebras of type F. Here we assume merely that the base field $\Phi$ is of characteristic 0. A Lie algebra $\mathfrak{S}$ is said to be of type F if
$\mathfrak{f}$ is the Lie algebra $F_4$ over $\Omega$ where $\Omega$ is the algebraic closure of $\Phi$. Our determination of the Lie algebras of type $F$ is given in terms of the exceptional central simple Jordan algebras over $\Phi$ defined in §1.

In [6] Jacobson characterizes Lie algebras of type $G$ as the derivation algebras of Cayley algebras over $\Phi$. The next three theorems in this paper are restatements of analogous theorems for algebras of type $G$. The statements made in §1 about exceptional simple Jordan algebras together with Theorem 1 allow us to use Jacobson's proofs. We shall merely give a brief outline of the proofs of Theorems 2 and 4.

**Theorem 2.** Let $\mathfrak{f}_1$ and $\mathfrak{f}_2$ be exceptional central simple Jordan algebras over a field $\Phi$ of characteristic 0 such that $\mathfrak{D}(\mathfrak{f}_1) \cong \mathfrak{D}(\mathfrak{f}_2)$. Then there exists a unique isomorphism $S$ between $\mathfrak{f}_1$ and $\mathfrak{f}_2$ such that the given isomorphism between $\mathfrak{D}(\mathfrak{f}_1)$ and $\mathfrak{D}(\mathfrak{f}_2)$ has the form $D \mapsto E = SDS^{-1}$.

Let $\Omega$ be the algebraic closure of $\Phi$. The algebras $\mathfrak{f}_1$ and $\mathfrak{f}_2$ may be regarded as subrings of $\mathfrak{f}$, the unique exceptional simple Jordan algebra over $\Omega$. The isomorphism between $\mathfrak{D}(\mathfrak{f}_1)$ and $\mathfrak{D}(\mathfrak{f}_2)$ may be extended to an automorphism of $\mathfrak{D}(\mathfrak{f})$. By our lemma there is a linear transformation $S_1$ of $\mathfrak{f}_0$ such that the given mapping of $\mathfrak{D}$ has the form $D \mapsto S_1DS_1^{-1}$ on $\mathfrak{f}_0$. Using bases of $\mathfrak{f}_0$ as bases of $\mathfrak{f}_0$, the matrix of $S_1$ may be taken with elements in $\Phi$ and moreover $S_1$ maps $\mathfrak{f}_0$ onto $\mathfrak{f}_0$. By the proof of Theorem 1 there is a unique automorphism $S$ of $\mathfrak{f}$ which maps $\mathfrak{f}_1$ onto $\mathfrak{f}_2$ and such that the isomorphism between $\mathfrak{D}(\mathfrak{f}_1)$ and $\mathfrak{D}(\mathfrak{f}_2)$ is given by $D \mapsto SDS^{-1}$.

**Theorem 3.** If $\mathfrak{f}$ is an exceptional central simple Jordan algebra over a field $\Phi$ of characteristic 0, then the group of automorphisms of $\mathfrak{D}(\mathfrak{f})$ is isomorphic to the group of automorphisms of $\mathfrak{f}$.

**Theorem 4.** A necessary and sufficient condition that a Lie algebra $\mathfrak{g}$ over a field $\Phi$ of characteristic 0 be of type $F$ is that $\mathfrak{g} \cong \mathfrak{D}(\mathfrak{f})$, $\mathfrak{f}$ an exceptional central simple Jordan algebra over $\Phi$.

Let $\mathfrak{g}$ be a Lie algebra of type $F$ over $\Phi$ and $\mathfrak{f}$ the unique exceptional Jordan algebra over $\Omega$, the algebraic closure of $\Phi$. The basal elements of $\mathfrak{g}$ may be represented as derivations $D_\alpha$ of $\mathfrak{f}$. If $(e_i)$ is a basis of $\mathfrak{f}$ and $D_\alpha e_i = \sum \gamma_\alpha^{(k)} e_k$, the $\gamma_\alpha^{(k)}$ may be taken as elements in a finite Galois extension $P$ of $\Phi$ such that $\mathfrak{g}_P$ is isomorphic to the derivation algebra of $(e_i)$ over $P$. Using Theorem 2 it may be shown that there is a $(1,1)$ representation of the Galois group of $P$ over $\Phi$ by semi-linear transformations of $(e_i)$ over $P$ which commute with the $D_\alpha$. Thus the conditions of the lemma of [6, p. 782] are satisfied and the set of elements invariant under these semi-linear transformations is a vector space of dimension 27 over $\Phi$. This space is closed with...
respect to multiplication and is a central simple Jordan algebra over \( \Phi \). The \( D_k \) are derivations of this algebra and the theorem is proved.

**Remark.** As a corollary to Theorems 2 and 4 we may remove the assumption of algebraic closure in Theorem 1: if \( D \rightarrow D^S \) is an automorphism of \( \mathcal{F} = D(\mathfrak{J}) \), any Lie algebra of type F over \( \Phi \) of characteristic 0, then there is a unique automorphism \( S \) of \( \mathfrak{J} \) such that \( D^S = SDS^{-1} \).

**Theorem 5.** A Lie algebra \( \mathcal{F} \) over a field \( \Phi \) of characteristic 0 is simple with multiplication center \( \mathfrak{P} \) and of type F over \( \Phi \) if and only if \( \mathcal{F} = D(\mathfrak{A}) \) for some exceptional simple Jordan algebra \( \mathfrak{A} \) with center \( \mathfrak{P} \).

If \( \mathfrak{A} \) is an exceptional simple Jordan algebra over \( \Phi \) with center \( \mathfrak{P} \), then \( \mathfrak{A} \) is central simple over \( \mathfrak{P} \). Since \( \Phi \) is of characteristic 0, the elements of \( \mathfrak{P} \) are such that \( DP = 0 \) for all \( D \) in \( D(\mathfrak{A}) \) [5]. Thus \( D(\mathfrak{A}) \) may be regarded as an algebra over \( \mathfrak{P} \), since \( D(\rho x) = \rho (Dx) \) for \( \rho \) in \( \mathfrak{P} \), \( x \) in \( \mathfrak{A} \). Therefore \( D(\mathfrak{A}) \) over \( \mathfrak{P} \) = \( D(\mathfrak{A} \text{ over } \mathfrak{P}) \) or \( D(\mathfrak{A}) \) is a Lie algebra of type F over \( \mathfrak{P} \).

Conversely, by Theorem 4, \( \mathcal{F} \) over \( \Phi \) is \( D(\mathfrak{A} \text{ over } \mathfrak{P}) \) where \( \mathfrak{A} \) over \( \mathfrak{P} \) is an exceptional central simple Jordan algebra. Over \( \Phi \), \( \mathfrak{A} \) is an exceptional simple Jordan algebra. Thus \( \mathcal{F} \cong D(\mathfrak{A}) \), since \( D\Phi = 0 \).

**5. Lie algebras of type F over a real closed field.** Let \( \Phi \) be a real closed field. It is known [3] that there are three nonisomorphic Lie algebras of type F over \( \Phi \). Then our Theorems 2 and 4 imply that there are three non-isomorphic exceptional central simple Jordan algebras \( \mathfrak{J} = \mathcal{H}(\mathfrak{C}, \rho) \) over \( \Phi \).

Let \( \mathfrak{C}_0 \) be the Cayley algebra with divisors of zero over \( \Phi \), \( \mathfrak{C}_1 \) be the Cayley division algebra over \( \Phi \), and

\[
\rho_0 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{pmatrix} \in \Phi_3.
\]

**Theorem 6.** The three nonisomorphic exceptional central simple Jordan algebras over a real closed field \( \Phi \) are

\[
\mathcal{H}(\mathfrak{C}_0, 1), \quad \mathcal{H}(\mathfrak{C}_1, 1), \quad \mathcal{H}(\mathfrak{C}_1, \rho_0)
\]

and the three Lie algebras of type F over \( \Phi \) are their derivation algebras.

For Jacobson has recently proved\(^{a}\) that \( \mathcal{H}(\mathfrak{C}_0, \rho) \cong \mathcal{H}(\mathfrak{C}_0, 1) \) for any

\(^{a}\) In a letter to Schafer dated 2/17/52 Jacobson remarks that he has proved \( \mathcal{H}(\mathfrak{C}_0, \rho) \cong \mathcal{H}(\mathfrak{C}_0, 1) \) for any \( \rho \) where \( \mathfrak{C}_0 \) is the unique Cayley algebra with divisors of zero over an arbitrary \( \Phi \).
\( \tilde{\rho} \). Let \( \mathcal{C} \) be any Cayley algebra over an arbitrary \( \Phi \). Let \( \tilde{\rho} = \alpha \rho g \rho' \) for \( \alpha \neq 0 \) in \( \Phi \), \( g \) nonsingular in \( \Phi \); that is, \( \tilde{\rho} \) differs from a matrix congruent to \( \rho \) by only a nonzero scalar factor. Then it is easy to see that \( \mathcal{H}(\mathcal{C}, \rho) \cong \mathcal{H}(\mathcal{C}, \tilde{\rho}) \) under the mapping \( x \rightarrow gxg^{-1} \). Over a real closed field \( \Phi \), any \( \rho \) is congruent to one of

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad -1.
\]

Hence any \( \mathcal{J} = \mathcal{H}(\mathcal{C}, \rho) \) over a real closed field \( \Phi \) is isomorphic to either \( \mathcal{H}(\mathcal{C}, 1) \) or \( \mathcal{H}(\mathcal{C}, \rho_0) \). Combining these results we see that \( \mathcal{J} \) is isomorphic to one of the algebras (23). But three nonisomorphic algebras do exist, and so these are the algebras (23).

References


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