THE EXISTENCE OF LINE INVOLUTIONS OF ORDER GREATER THAN THREE POSSESSING A LINEAR COMPLEX OF INVARIANT LINES

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Introduction. In a recent paper [1] attention was called to a new family of line involutions in \( S_3 \) furnishing examples of involutions of all orders, \( m, \geq 4 \) with complexes of invariant lines of all possible orders, \( i \), from 2 up to the maximum, \( [(m+1)/2] \). Since involutions of all orders without a complex of invariant lines are known to exist, and since examples of all possible involutions of order \(<4\) are known, the only involutions for which existence examples remain to be supplied are those whose orders are \( \geq 4 \) and whose invariant lines form a linear complex. It is the purpose of this note to define a class of involutions having these properties, thus establishing the existence of line involutions corresponding to every admissible set of characteristics \((m, n, i, k)\) [1]. In our development we shall work exclusively on the nonsingular \( V^2_4 \) in \( S_5 \) into whose points the lines of \( S_3 \) are mapped in a 1:1 way by the well known interpretation of the Plücker coordinates of a line in \( S_4 \) as point coordinates in \( S_5 \).

1. The definition of the involution. On \( V^2_4 \) let there be given a point \( O \) and a plane \( \pi \) in general position, and let \( \lambda \) be the line in which the tangent hyperplane to \( V^2_4 \) at \( O \) meets \( \pi \). Moreover, let \( C_\alpha \) be a curve of order \( \alpha \) lying on \( V^2_4 \) and meeting \( \pi \) in \( \alpha - 1 \) points, \( \beta \) of which fall on the line \( \lambda \). Obviously \( C_\alpha \) lies in an \( S_4 \) and is rational. Finally let \( \Gamma \) be the set of points on \( V^2_4 \) which represents a general linear complex of lines in \( S_4 \); in other words, \( \Gamma \) is the intersection of \( V^2_4 \) and a general \( S_4 \).

Now if \( P \) is a general point of \( V^2_4 \) (the image of a general line in \( S_4 \)), then \( PO\pi \) is a 4-space which meets \( C_\alpha \) in the \( \alpha - 1 \) fixed points common to \( C_\alpha \) and \( \pi \), and in one additional point \( Q \) which varies with \( P \). The point \( Q \) will of course coincide with one of the fixed intersections of \( C_\alpha \) and \( \pi \) if and only if \( PO\pi \) is a 4-space which contains the tangent to \( C_\alpha \) at one of these intersections. Thus, there is a unique plane \( \sigma = POQ \) which passes through \( P \) and \( O \), meets \( \pi \) (since it lies in an \( S_4 \) with \( \pi \)), and intersects \( C_\alpha \) in a point distinct (in general) from the \( \alpha - 1 \) intersections of \( C_\alpha \) and \( \pi \). Now \( \sigma \) meets \( V^2_4 \) in a conic, \( \gamma \), and this conic meets the given linear complex \( \Gamma \) in two points, say \( M \) and \( N \). Finally, \( P \) (which of course lies on \( \gamma \)) has a unique harmonic con-
jugate, $P'$, with respect to $M$ and $N$ on the conic $\gamma$. We consider the transformation that takes $P$ to $P'$; this is obviously involutory. If $P$ is distinct from $M$ and $N$, i.e., if $P$ is not in the linear complex $\Gamma$, then $P$ and $P'$ are necessarily distinct, and so $P$ cannot be invariant. On the other hand, if $P$ is in $\Gamma$, then on $\gamma$, $P$ coincides either with $M$ or with $N$ and, from the elementary properties of harmonic ranges, $P'$ must coincide with $P$, i.e., $P$ is an invariant point. Hence, the points of $\Gamma$, and only those points, are invariant. Thus, the involution has a linear complex of invariant elements, as desired, and it remains to determine the order, $m$, of the involution and verify that it can take on any value $\geq 4$.

2. The order of the involution. To determine $m$ it is convenient to solve first for the number, $k$, of points on a general line $l$ of $V_4^\delta$ which are singular, i.e. have the property that the line $PP'$ lies entirely on $V_4^\delta$. Then we can find $m$ at once from the formula $m = k + 2i - 1$ of [1]; in fact, since $i = 1$ in the present case, $m = k + 1$.

To find $k$ we observe first that the line joining a point $P$ to its image $P'$ will lie entirely on $V_4^\delta$ if and only if the plane $\sigma$ determined by $P$ meets $V_4^\delta$ in a conic consisting of a pair of lines. Moreover, when this is the case, one of the lines must pass through $O$.

Now consider a general line $l$ on $V_4^\delta$. From the nature of the involution it is evident that the points of $l$ are in 1:1 correspondence with the points of $C_{a}$. On $l$ there are three and only three classes of points for which $\sigma$ meets $V_4^\delta$ in a pair of lines. These arise respectively when the line of $V_4^\delta$ which passes through $O$ in $\sigma$:

1. meets $l$,
2. meets $C_{a}$,
3. meets $\pi$ in a point distinct from any of the $\beta$ intersections of $\lambda$ and $C_{a}$.

In Case 1, the singular point, $L$, on $l$ is unique, being in fact the intersection of $l$ and the $S_{a}$ which is tangent to $V_4^\delta$ at $O$.

In Case 2, we note that the $S_{a}$ which is tangent to $V_4^\delta$ at $O$ meets $C_{a}$ in $\alpha$ points, consisting of the $\beta$ points $Q_{j}^{(1)}$ common to $C_{a}$ and $\lambda$, and $\alpha - \beta$ additional points, $Q_{j}^{(0)}$, which are not in $\pi$. The line joining each of these points to $O$ obviously lies entirely on $V_4^\delta$. Now $\pi$ and the line joining $O$ to any of the $\alpha - \beta$ points $Q_{j}^{(0)}$ determine an $S_{a}$ which meets $l$ in a point, say $L_{j}^{(0)}$. Similarly, $\pi$, the line joining $O$ to any one of the $\beta$ points $Q_{j}^{(1)}$, and the tangent to $C_{a}$ at $Q_{j}^{(0)}$, determine an $S_{a}$ which meets $l$ in a point, say $L_{j}^{(0)}$. Any of the points $L_{j}^{(1)}$ and $L_{j}^{(0)}$ determine with $O$ and the corresponding point $Q_{j}^{(1)}$ or $Q_{j}^{(0)}$ a plane $\sigma$ which passes through $O$, meets $C_{a}$ and $\pi$, and intersects $V_4^\delta$ in a com-
posite conic. Moreover, the $L_j^{(1)}$ and $L_j^{(2)}$ are clearly all distinct, and different from the point $L$ obtained in Case 1. Thus the $L_j^{(1)}$ and $L_j^{(2)}$ constitute $\alpha$ additional singular points on $l$.

Finally, in Case 3, there is a unique plane of $V_4^2$ passing through $O$ and meeting $\pi$ in a line, namely the plane of $\sigma\lambda$. This plane and $l$ determine an $S_4$ which meets $C_\alpha$ in $\alpha$ points, $R_j$, including the $\beta$ intersections of $C_\alpha$ and $\lambda$ which we have already taken into account, and hence now reject. In this $S_4$, the 3-space $lOR_j$ meets $\lambda$ in a point, say $G_j$. Moreover, since $l$ and the plane $\sigma = OR_jG_j$ lie in the same 3-space, $\sigma$ meets $l$ in a point, say $L_j$. Since $\sigma$ clearly passes through $O$, meets $l$, $C_\alpha$, and $\pi$, and intersects $V_4^2$ in a composite conic (consisting of the lines $OG_j$ and $L_jR_j$) the points $L_j$ are also singular points on $l$. Further, it is clear that the $L_j$ are all distinct and different from $L$ and any of the points $L_j^{(1)}$ and $L_j^{(2)}$. Hence they constitute $\alpha - \beta$ additional singular points in the set of singular points on $l$ which we are enumerating.

Therefore, on $l$ we have altogether

$$k = 1 + \alpha + (\alpha - \beta) = 2\alpha + \beta + 1$$

singular points. Hence, the order of the involution is

$$m = k + 1 = 2\alpha + \beta + 2 \quad (\beta \leq \alpha - 1).$$

Thus beginning with $\alpha = 1$ and $\beta = 0$ we can obtain involutions of all orders $\geq 4$ possessing linear complexes of invariant elements.

**Reference**


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