LATTICE PACKING IN THE PLANE WITHOUT CROSSING ARCS
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Introduction. We first review some definitions and results of Chalk and Rogers. If $S$ and $T$ are two sets of points in Euclidean $n$-space, then $S+T$ will denote the set of all points $s+t$ where $s$ is in $S$ and $t$ is in $T$, while $S-T$ is composed of all $s-t$. The point set sum will be denoted by $S\cup T$ and the intersection by $S\cap T$. Let $\Lambda$ be a lattice; then $S+\Lambda$ is a lattice packing if no two sets $S+\lambda$ and $S+\lambda'$, with $\lambda$ and $\lambda'$ distinct points in $\Lambda$, have a common point in their interiors. Let $D(S)$ be the set of all $s-s'$ where $s$ and $s'$ are points in the interior of $S$. Chalk and Rogers have shown that $S+\Lambda$ is a lattice packing if and only if the lattice $\Lambda$ is admissible for $D(S)$, i.e., has no point in the interior of $D(S)$ except possibly the origin $O$. If $S$ is convex this criterion reduces to a result of Minkowski, well known especially when $S$ is symmetric. For, if $S$ is also open, then $D(S)=S-S=S+S=2S$, this being the set $S$ expanded by a factor 2.

From now on we assume that all point sets lie in the plane. We wish to provide a similar criterion for the situation in which no arc of $S+\lambda$ crosses an arc of $S+\lambda'$. Then $S+\Lambda$ is called a lattice packing without crossing arcs, and is in particular a lattice packing.

Since $D(S)$ omits from consideration all arcs of $S$ not in the interior of $S$, it is to be expected that it will be of no use for our purpose. Our criterion will refer instead to the set $E(S)$ defined as $S-S$. Clearly $E(S)$ is symmetric. It is easy to see that $E(S)$ is generated by translating $S$ such that always one of its points is at $O$. If $E(S)$ is a circle and its interior then $S$ is a figure of constant breadth and conversely.

If $S$ is the square $0 \leq x \leq 1$, $0 \leq y \leq 1$, and $T$ consists of three sides of this square, omitting the side on the $y$-axis, then $E(S)=E(T)$. Yet there are lattices which provide lattice packings without crossing arcs for $T$ but which do not do so for $S$; e.g., the lattice generated by $(1/2, 0)$ and $(0, 1)$. The condition we obtain will involve local properties of $E(S)$ arising from $S$.
We first discuss the crossing of arcs in general and obtain criteria applicable to our problem.

1. **Crossing arcs.** Denote by $e(A)$ the point set consisting of the two end points of an arc $A$.

**Definition.** Let $A, B$ be arcs with a common subarc $M$. Suppose that for every neighborhood $V \supset M$ there are arcs $A', B' \subset V$ and an arc $D$ satisfying the following conditions.

(i) $A \supset A' \supset M; B \supset B' \supset M; e(D) = d(B');$

(ii) $D \cap (A' \cup B') = e(B');$

(iii) $A$ meets the interior of the closed curve $B' \cup D$ in a point $a_1$ and the exterior at a point $a_2$.

Then we say that $A$ *V-crosses* $B$. If also

(iv) $e(B') \cap A' = \emptyset$

then we say that $A$ *crosses* $B$; and if in addition

(v) $e(A') \cap B' = \emptyset,$

we say that $A$ *s-crosses* $B$.

If $M$ consists of a single point, we speak of a point-crossing, point- $V$-crossing, or point-s-crossing.

Of these only the s-crossing is symmetric. An example of $A$ crossing $B$ but $B$ not crossing $A$ occurs when $A$ is the $x$-axis from $-1$ to $2$, $M$ is the $x$-axis from $0$ to $1$, and $B$ is $M$ and $y = x \sin 1/x (-1 \leq x < 0)$.

**Lemma 1.** If $A$ s-crosses $B$ at $M$, then $B$ s-crosses $A$ at $M$.

**Proof.** Let $A', B', a_1, a_2, D$ be as in the definition. Join the end points of $A'$ by an arc $D_2$ which meets $A'$ nowhere else. Then $D_2 \cup A'$ is a simply closed curve. Now $B' \cap D_2 \cap A' = \emptyset$. Hence there is a subarc $B_1$ of $B$ such that $M \subset B_1, B_1 \cap D_2 = \emptyset$, and $e(B_1) \cap A' = \emptyset$. Such a subarc $B_1$ may be obtained by taking any subarc in the interior of the largest subarc $B_3$ of $B$ containing $M$ whose interior is disjoint from $D_2$, but itself containing in its interior the largest subarc of $B_3$ with end points in $A'$.

If $B_1$ has points both interior and exterior to $D_2 \cup A'$, then $B$ s-crosses $A$ at $M$. Otherwise suppose that no point of $B_1$ is interior to $D_2 \cup A'$.

Choose a neighborhood of $M$ sufficiently small that its boundary meets $A'$ in points separated by $M$ and likewise for $B'$. In $W$ there are arcs $A'_1, B'_1, D_1$ satisfying the definition of s-crossing. Necessarily $A'_1 \subset A', B'_1 \subset B'$; let $a'_1$ be a point of $A'_1$ lying interior to $B'_1 \cup D_1$, etc.
and $a'_2$ a point of $A'_1$ exterior. Let $\epsilon$ be the smaller of the distance of $A'_1$ from $D_1$ and of $a'_1$, $a'_2$ from $B'_1 \cup D_1$; thus $\epsilon > 0$. Choose points $p_1$, $p_2$ interior to $D_1 \cup A'$ but with the distance of $p_1$ to $a'_1$ and of $p_2$ to $a'_2$ less than $\epsilon$. Thus $p_1$ is interior to $B'_1 \cup D_1$ and $p_2$ exterior. Join $p_1$ to $p_2$ by an arc $E$ lying interior to $D_2 \cup A'$ and every point of which is less than distance $\epsilon$ from $A'_1$. Then $E$ does not intersect $D_1$, but has points interior and exterior to $D_1 \cup B'_1$. Hence it intersects $B'_1$ and a fortiori $B'$. But since $E$ lies interior to $D_2 \cup A'$ it cannot intersect $B'$. This contradiction shows that the assumption that $B_1$ had no points interior to $D_1 \cup A'$ is false.

A very similar argument shows that we cannot assume that $B_1$ has no point exterior to $D_2 \cup A'$. Thus $B_1$ has points both interior and exterior to $D_2 \cup A'$. Hence $B_1$ s-crosses $A$ at $M$.

**Lemma 2.** If $A$ does not cross $B$ at $M$, then $A$ V-crosses $B$ at $M$ if and only if $A$ crosses $B$ in every neighborhood of $M$.

The proof for "if" is immediate.

Conversely, suppose $A$ V-crosses, but does not cross, $B$ at $M$. Then for some neighborhood $V$ of $M$ the points $a_1$, $a_2$ as given in the definition of V-crossing will be the end points of a subarc $A_1$ of $A$ which does not contain $M$. In going from $a_1$ to $a_2$ on $A_1$ let $m$ be the last point which lies on $B$ such that no preceding point lies exterior to $B' \cup D$, and let $M_1$ be the component of $A'_1 \cap B'$ containing $m$. This component $M_1$ is contained in the interior of $B'$ since $e(B') \cap A = \emptyset$. Also $M_1$ is contained in the interior of $A_1$ since it lies between $a_1$ and $a_2$ and does not contain them.

We show finally that $A$ crosses $B$ at $M$. Let $V_1$ be a neighborhood of $M_1$. By our construction the component of $B \cap V_1$ which contains $M_1$ has points $b_1$, $b_2$ separated by $M_1$ on $B$ and not in $A$. Let $B'_1$ be the subarc of $B$ with end points $b_1$, $b_2$. Let $D_1$ be the rest of the simple closed curve $B \cup D$; then $B_1 \cup D_1$ and $B \cup D$ are the same simple closed curves. Let $r_1$ be the distance of $M_1$ from $D_1$; hence $r_1 > 0$. Let $V$ be the set of points whose distance from $M_1$ is less than $r_1$, and let $A'_1$ be the component of $M_1$ in $A \cap V$ ($V$ denotes the closure of $V$). Then by the construction $A'$ contains points $a'_1$, $a'_2$ separated by $M_1$ and not in $B$, where $a'_2$ is exterior to $B \cup D$ and $a'_1$ is of necessity interior. Then in the definition of $A$ crossing $B$ at $M_1$ use $V_1$, $A'_1$, $B'$, $a'_1$, $a'_2$.

**Theorem 1.** If an arc $A$ crosses an arc $B$, then $B - A$ contains a neighborhood of the origin.

Let $A'$, $B'$, $D$, $a_1$, and $a_2$ be as described in the definition. Let $A''$
be the subarc of $A'$ joining $a_1$ and $a_2$. Let $r$ be the smallest of the following distances: of $a_1$ from $B' \cup D$, of $a_2$ from $B' \cup D$, and of $A''$ from $D$. Translate $A''$ by an amount $r' < r$ in any direction $\theta$ to obtain an arc $A_1$; then $A_1$ cannot meet $D$. Now $a_1$ goes to a point $a'_1$ and $a_2$ to $a'_2$; also $a'_1$ is interior to $B' \cup D$ while $a'_2$ is exterior. Hence $A_1$ intersects $B' \cup D$ and must do so in $B'$ and at a point $b$ which is the image under the translation of a point $a$ of $A$. The point $b - a$ has polar coordinates $(r', \theta)$. Thus $B - A$ contains the interior of the circle of radius $r$ and with center at the origin.

The converse of the theorem is not true. As a counterexample let $A$ and $B$ both be the spiral given in polar coordinates by $r = 1/\theta^2$ ($1 \leq \theta \leq \infty$) and $M$ be the origin. Another counterexample is given by letting $A$ be the double spiral $r = 1/\theta^3$ and $B$ the double spiral $r = 1/\theta^2 + \pi/2$ and $B$ the double spiral $r = 1/\theta + 3\pi/2$, where throughout $1 \leq \theta \leq \infty$.

Yet another counterexample occurs when $A$ and $B$ are the same triod $S$; i.e., $S$ consists of three arcs, having an end point $p$ in common but no other point common to any two of them.

**Lemma 3.** If $S$ is a triod then $E(S)$ contains a neighborhood of the origin.

Suppose that $S$ consists of the three arcs $A$, $B$, $C$, each having $p$ as an end point and no two having any other point in common. Let the other end points be $a$ on $A$, $b$ on $B$, and $c$ on $C$. Let $D$ be a simple closed curve containing $a$, $b$, $c$ but no other points of $A$, $B$, $C$, and with $p$ inside $D$. Let $D_{ab}$ be the subarc of $D$, with end points $a$ and $b$ and not containing $c$; $D_{bc}$ and $D_{ca}$ are similarly defined. Then $D = D_{ab} \cup D_{bc} \cup D_{ca}$. Also $D_{ab} \cup A \cup B$ is a simple closed curve.

The remainder of the proof is similar to that of Theorem 1. Let $a'$ be a point on the interior of $A$ and let $A'$ be the subarc of $A$ joining $p$ and $a'$. Define $b'$, $B'$ and $c'$, $C'$ similarly. Let $r_a$ be the lesser of the distances of $a'$ from $B \cup C$ and $A'$ from $D$, with corresponding definitions for $r_b$ and $r_c$. Take $r$ as the minimum of $r_a$, $r_b$, $r_c$. Then $r > 0$.

Translate $F = A' \cup B' \cup C'$ by an amount $r' < r$ in any direction $\theta$, obtaining the set $F_1$. Then $p$ goes into a point $p_1$ which must lie inside $D$. But $p_1$ must be inside or on the boundary of one of the simple closed curves $D_{ab} \cup A \cup B$, $D_{bc} \cup B \cup C$, or $D_{ca} \cup C \cup A$; assume it is so for the first curve. Then the image $c'_1$ of $c'$ lies outside that simple closed curve and the image $C'_1$ of $C'$, having one end point outside and other inside or on the boundary of $D_{ab} \cup A \cup B$, must intersect it at a point $w$ which is also the image $z_1$ of a point $z$ on $C'$. This intersection $w$ cannot be on $D$; hence it lies on $A \cup B$. Thus $w - z = z_1 - z$. 

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is the point with polar coordinates \((r', \theta)\) and lies in \(E(S)\). This shows that \(E(S)\) contains the circle of radius \(r\) and center at 0, as was to be proved.

**Theorem 2.** Let \(B\) be an arc for which \(E(B)\) has no point with vectorial angle \(\theta\) except the origin, i.e., no secant of \(B\) has inclination \(\theta\). Suppose \(M\) is a subarc in the interior of \(B\) and also of an arc \(A\). Then \(A\) does not cross \(B\) at \(M\) if and only if there are a neighborhood \(V\) containing \(M\) and a direction \(\alpha\) such that no point in \((B \cap V) - (A \cap V)\) has vectorial angle \(\alpha\) except the origin.

The "if" part is a consequence of the previous theorem.

If \(A\) does not cross \(B\) at \(M\) there is a subarc \(A'\) of \(A\) and a subarc \(B'\) of \(B\), both containing \(M\) but with end points different from those of \(M\), and an arc \(D\) joining the end points of \(B'\) and not otherwise intersecting \(A'\) or \(B'\) such that no point of \(A'\) lies in the interior of the simple closed curve \(B' \cup D\). Choose a subarc \(B''\) of \(B'\) containing \(M\) but with end points distinct from those of \(B'\). Let \(d\) be the distance of \(B''\) from \(D\). Let \(W\) be the set of points with polar coordinates \((d', \theta), 0 < d' < d/2\). None of the points of \(B'' + W\) lies on \(D\) since \(d' < d/2\) and none lies on \(B\) by the hypothesis on \(\theta\). The same is true of \(B'' - W\). Thus either \(B'' + W\) or \(B'' - W\) lies in the interior of \(B' \cup D\); let it be denoted by \(B + W\). Let \(\alpha = \theta + \pi\) or \(\theta\) according as \(W = W_1\) or \(-W_1\). Then \(B'' - W\) lies interior to \(B' \cup D\) and hence cannot intersect \(A'\). Thus \(B'' - A'\) does not contain \(W\). Finally, if \(V\) is taken such that its distance from \(M\) is \(<d/2\), we see that the theorem is satisfied.

2. **Local boundary point.** Before returning to the problem of lattice packings, we introduce another term.

**Definition.** A point \(p\) is called a local boundary point of \(A + B\) if

- (1) \(p\) lies in the closure \(\text{Cl} (A + B)\) of \(A + B\), and
- (2) whenever \(p = a + b\) with \(a\) in \(A\) and \(b\) in \(B\), there exist neighborhoods \(U\) of \(a\) and \(V\) of \(b\) such that \(p\) is a boundary point of \((A \cap U) + (B \cap V)\).

By a local boundary point of \(E(S)\) is meant a local boundary point of \(S + (-S)\).

If \(S\) is composed of three sides of a square, then \(E(S)\) is a square and the locus of its local boundary points consists of the four sides and also the two lines joining the midpoints of opposite sides.

**Lemma 4.** If \(A\) is open, then \(A + B\) contains none of its local boundary points.

For if \(p = a + b\) with \(a\) in \(A\) and \(b\) in \(B\), then for every neighborhood \(U\) of \(a\) with \(U \subset A\), \(U + b\) is a neighborhood of \(p\) and \(U + b \subset A + B\).
Hence for any neighborhood $V$ of $b$, $(U \cap A) + (V \cap B) = U + (V \cap B)$ contains the neighborhood $U + b$ of $p$ since $b$ is in $V \cap B$. Thus $p$ cannot be a local boundary point.

The converse, that if $A + B$ contains none of its local boundary points then either $A$ or $B$ is open, is not true. Counterexample: $A = B$ and is the punctured disc consisting of every point whose distance $d$ from the origin satisfies $1 \leq d < 4$.

3. Lattice packings.

Theorem 3. In order that $S + \Lambda$ be a lattice packing without point-crossing arcs it is sufficient that the points of $\Lambda$ distinct from 0 and belonging to $E(S)$ be local boundary points of $E(S)$.

The proof is by contradiction. Suppose that there are arcs $A$ and $B$ in $S$ such that $A + \lambda'$ crosses $B + \lambda''$, where $\lambda'$ and $\lambda''$ are distinct points of $\Lambda$. Then $A$ and $B + \lambda$ cross, where $\lambda = \lambda'' - \lambda'$. Hence by Theorem 1, $A - (B + \lambda)$ contains a neighborhood of 0 and $A$ and $B + \lambda$ may be be taken to lie in any neighborhood, no matter how small, of the point of crossing. Thus $A - B$ contains a neighborhood of $\lambda$. Also $A - B$ lies in $E(S)$. Hence $\lambda$ is not a local boundary point of $E(S)$. But $\lambda$ is a point of $\Lambda$ distinct from 0. The existence of such a point is a contradiction to the hypothesis on the nonexistence of such points, because of Lemma 2.

If in Theorem 3 crossings on common subarcs, not simply points, are to be excluded, then the following modification needs to be made. Whenever two subarcs $A$, $B$ of $S$ can be made to coincide by a translation, i.e., $A = B + p$ for some point $p$, then in determining whether $p$ or $-p$ is a local boundary point only those neighborhoods of $A$ or $B$ are used which include maximal arcs $A_1$ or $B_1$ such that $A_1 \subset A$, $B_1 \subset B$, and $A_1 = B_1 + p$.

Although Theorem 2 provides a partial converse to Theorem 1, yet a partial converse of Theorem 3 cannot be obtained from Theorem 2 without imposing rather stringent hypotheses.

An example illustrating some of the difficulty is the following. On the interval $-\epsilon \leq x \leq \epsilon$, let $S$ be the set of all horizontal line segments with $y$ rational and between $-\epsilon$ and $\epsilon$ and let $T$ be the same except that $y$ is irrational or 0. Then $S - T$ contains a neighborhood of the origin and the origin is not a local boundary point, although $S$ and $T$ do not have crossing arcs.

Theorem 2 could be used if one required that on every arc $A$ of $S$ every point has a neighborhood in which the secants of $A$ do not have all directions and by modifying the definition of a local boundary...
point so that instead of \( U \cap A \) and \( V \cap B \) one takes any subarc of \( A \) in \( U \) containing \( a \) and any subarc of \( B \) in \( V \) containing \( b \). These changes are felt to be undesirable in being too great a departure from the original concept of \( D(S) \) and in excluding sets with nonempty interiors.

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**THE DEGREE FORMULA FOR THE SKEW-REPRESENTATIONS OF THE SYMMETRIC GROUP**

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1. Introduction. In his paper on the representations of the symmetric group,\(^2\) G. de B. Robinson defines certain "skew-representations" and associates these to skew-diagrams (to be defined below) analogously to the way the irreducible representations of the symmetric group are associated with regular diagrams. Furthermore he shows that the degree of such a skew-representation is equal to the number of orderings of the related skew-diagram.\(^3\)

The object of this note is to derive a formula for the degree of skew-representation related to a given skew-diagram.\(^4\) This problem will be treated strictly in terms of the number of orderings of such a diagram, and from this point of view is very similar to the question attacked in [5] by R. M. Thrall.

In §4, this formula is applied to the problem of computing the characters of certain classes of the symmetric group.

2. Definitions and lemmas. A partially ordered set \( P \) is said to be regular or a regular diagram if:

(I) The elements of \( P \) may be represented by ordered pairs of integers \((i, j), i > 0, j > 0, \) where \((i, j) \leq (p, m)\) if and only if \( i \leq p \) and \( j \leq m \), \((i, j) = (p, m)\) if and only if \( i = p \) and \( j = m \),

(II) \( \max_i (i, j) \leq \max_i (i, j') \) whenever \( j \geq j' \),

(III) \((i, k) \in P\) implies \((j, k) \in P\) for all integers \( j \) with \( 1 \leq j \leq i \),

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\(^{1}\) The work on this paper was performed under the sponsorship of the O.N.R.

\(^{2}\) See [1; 2; 3; 4].

\(^{3}\) See [1, p. 290].

\(^{4}\) This is an answer to the question raised in [1, p. 294], the \( \phi \) of that paper is the \( g \) of the theorem below.