A SPECIAL CONGRUENCE

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1. It is familiar that if $p$ is a prime such that $p-1|m$, $p^r|m$ then

\[(1.1) \quad B_m \equiv 0 \pmod{p^r},\]

where $B_m$ denotes a Bernoulli number in the even suffix notation. The writer has recently proved the companion formula ([2, Theorem 3]; see also [1])

\[(1.2) \quad B_{m(p-1)} + 1/p - 1 \equiv 0 \pmod{p^r} \quad (p \geq 3),\]

for $p^r|m$, $m > 0$; moreover if $m = p^r n$, then

\[(1.3) \quad p^r (B_{m(p-1)} + 1/p - 1) \equiv w_p \pmod{p} \quad (p > 3),\]

where $w_p$ denotes Wilson’s quotient $((p-1)!+1)/p$.

In this note we show that the above formulas imply

\[(1.4) \quad \sum_{0<s<p-1<p} \binom{m}{s(p-1)} \equiv 0 \pmod{p^r},\]

where $p^r|m$ and $p \geq 3$. More precisely if $m = p^r n$, we have, for $p > 3$,

\[(1.5) \quad \sum_{0<s<p-1<p-1} \binom{m}{s(p-1)} B_{2s} + \delta_m \frac{w_p}{p-1} \pmod{p},\]

where $\delta_m = 1$ for $p-1|2s-1$, $\delta_m = 0$ otherwise.

For $r = 0$, (1.4) is due to Hermite. The proof below of (1.4) was suggested by Nielsen’s proof [3, p. 254] of Hermite’s formula.

2. Proof of (1.4). Using the basic recurrence for the Bernoulli numbers we may write

\[(2.1) \quad 1 - \frac{1}{2} m + \sum_{0<s<p} \binom{m}{s} B_{2s} = 0.\]

Now let $p^r|m$. Consider first a term such that $p-1|2s$. Let $p^k|s$, so that by (1.1), $B_{2s} \equiv 0 \pmod{p^k}$. If $k \leq r$, it follows that

\[(2.2) \quad \binom{m}{2s} = \frac{m}{2s} \binom{m-1}{2s-1} \equiv 0 \pmod{p^r}.\]

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and consequently

\[(2.3) \quad \binom{m}{2s} B_{2s} \equiv 0 \pmod{p^r}.\]

Clearly (2.3) holds also for \(k > r\). Thus (2.1) and (2.2) imply

\[1 + \sum_{0 < s (p-1) < m} \binom{m}{s(p - 1)} B_{s(p-1)} \equiv 0 \pmod{p^r},\]

which may be rewritten as

\[1 + \sum_{0 < s (p-1) < m} \binom{m}{s(p - 1)} \left( B_{s(p-1)} + \frac{1}{p} - 1 \right) \equiv \left( \frac{1}{p} - 1 \right) \sum_{0 < s (p-1) < m} \binom{m}{s(p - 1)} \pmod{p^r}.\]

Now exactly as in proving (2.3), we may show, using (1.2), that

\[\binom{m}{s(p - 1)} \left( B_{s(p-1)} + \frac{1}{p} - 1 \right) \equiv 0 \pmod{p^r}.\]

Thus (2.4) reduces to

\[(2.5) \quad 1 \equiv \left( \frac{1}{p} - 1 \right) \sum_{0 < s (p-1) < m} \binom{m}{s(p - 1)} \pmod{p^r}.\]

It is evident that (2.5) and (1.4) are equivalent.

3. Proof of (1.5). We again begin with (2.1) which we now write as

\[1 - \frac{1}{2} m + \sum_{0 < 2s < m, p-1 | 2s} \binom{m}{2s} B_{2s} + \sum_{0 < s (p-1) < m} \binom{m}{s(p - 1)} B_{s(p-1)} = 0.\]

This evidently implies

\[1 - \frac{1}{2} m + \sum_{0 < 2s < m, p-1 | 2s} \binom{m}{2s} B_{2s} + \sum_{0 < s (p-1) < m} \binom{m}{s(p - 1)} \left( B_{s(p-1)} + \frac{1}{p} - 1 \right) = \sum_{0 < s (p-1) < m} \binom{m}{s(p - 1)} \left( \frac{1}{p} - 1 \right).\]
Consider first the sum

\[(3.2) \quad S = \sum_{0 < s < (p-1) < m} \binom{m}{s(p-1)} \left( B_{s(p-1)} + \frac{1}{p} - 1 \right). \]

Let \( p^k \mid s \) and put

\[ s = p^k h; \]

then by (1.3) we have

\[(3.3) \quad B_{s(p-1)} + \frac{1}{p} - 1 = p^k h w_p \pmod{p^{k+1}}. \]

If \( k \leq r \) it is evident from (2.2) that (3.3) yields

\[(3.4) \quad \binom{m}{s(p-1)} \left( B_{s(p-1)} + \frac{1}{p} - 1 \right) = \binom{m}{s(p-1)} p^k h w_p \pmod{p^{r+1}}; \]

clearly (3.4) holds also for \( k > r \). Since the right member of (3.4) is equal to

\[ m \binom{m-1}{s(p-1)-1} w_p / (p-1), \]

we see that (3.2) becomes

\[(3.5) \quad S = \frac{m w_p}{p-1} \sum_{0 < s < (p-1) < m} \binom{m-1}{s(p-1)-1} \pmod{p^{r+1}}. \]

In the next place for the first sum in the left member of (3.1) we have

\[(3.6) \quad \sum_{0 < 2s < m, p-1 < 2s} \binom{m}{2s} B_{2s} = m \sum_{0 < 2s < m, p-1 < 2s} \binom{m-1}{2s-1} \frac{B_{2s}}{2s}. \]

Substituting from (3.5) and (3.6) in (3.1) we get

\[(3.7) \quad 1 - \left( \frac{1}{p} - 1 \right) \sum_{0 < s < (p-1) < m} \binom{m}{s(p-1)} \]

\[\equiv \frac{1}{2} \left[ m - m \sum_{0 < 2s < m, p-1 < 2s} \binom{m-1}{2s-1} \frac{B_{2s}}{2s} \right. \]

\[\left. - \frac{m w_p}{p-1} \sum_{0 < s < (p-1) < m} \binom{m-1}{s(p-1)-1} \pmod{p^{r+1}}. \right] \]
Now let $\varphi^r \mid m$, $\varphi^{r+1} \mid m$; then (3.7) becomes
\[
\frac{1}{m} \left\{ 1 - \left( \frac{1}{\varphi} - 1 \right) \sum_{0 < s(\varphi - 1) < m} \binom{m}{s(\varphi - 1)} \right\} = \frac{1}{2} - \sum_{0 < 2s < m, \varphi - 1 \mid 2s} \frac{B_{2s}}{2s} - w_p \frac{m - 1}{\varphi - 1} \sum_{0 < s(\varphi - 1) < m} \binom{m}{s(\varphi - 1) - 1} \pmod{\varphi}.
\]
(3.8)

But [3, p. 255]
\[
\sum_{0 < s(\varphi - 1) < m} \binom{m}{s(\varphi - 1) - 1} = \begin{cases} 0 & (\varphi - 1 \mid m - 1), \\ -1 & (\varphi - 1 \mid m - 1); \end{cases}
\]
(3.9)

indeed (3.9) is an easy consequence of the case $r = 0$ of (1.4). Finally (3.7), (3.8), and (3.9) evidently imply (1.5).

References